Previously, on Homological Algebra:

Key Example
R-mod

$\mathcal{A}$ is an abelian category if:

- $\mathcal{A}$ is additive $\left(\text{Hom}_{\mathcal{A}} \text{ are all abelian groups, } \circ \text{ bilinear, } \mathcal{A} \text{ has a zero object and all finite } \underset{\text{coproducts}}{\overset{\text{products}}{\text{|||}}} \right)$
- $\mathcal{A}$ has all kernels and cokernels
- mono $\Rightarrow$ kernels and epi $\Rightarrow$ cokernel
- the canonical arrow

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \\ \operatorname{coker}\ker f \arrow[r, dashed, "\text{iso}"] & \ker\operatorname{coker} f \arrow[u] \end{tikzcd}$$

is an iso

$\Downarrow$

Every arrow f factors as $f = \text{mono} \circ \text{epi}$

<u>Image</u> $\quad \operatorname{im} f = \ker \operatorname{coker} f$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \\ \operatorname{coker}\ker f \arrow[r, dashed, "\text{iso}"] & \ker\operatorname{coker} f \arrow[u, "\operatorname{im} f"'] \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "\text{epi}"'] & & B \\ & \operatorname{im} f \arrow[ur, "\text{mono}"'] & \end{tikzcd}$$

Given an abelian category $\mathcal{A}$:

- $Ch(\mathcal{A})$ is an abelian category
- $\mathcal{A}^{op}$ is abelian

(C, ∂) in $Ch(\mathcal{A})$

$$\underset{\text{cycles}}{Z_n(C)} := \overset{\text{object}}{\ker \partial_n} \qquad \qquad B_n(C) := \overset{\text{object}}{\operatorname{im} \partial_n}$$

boundaries

<u>Remark</u> $\mathcal{A}$ abelian category $C \xrightarrow{f} D \xrightarrow{g} E$ such that $gf = 0$

<u>Claim</u> there is a canonical arrow $\operatorname{im} f \longrightarrow \ker g$

$$\begin{tikzcd}
C \arrow[r, "f"] \arrow[d, dashed, "epi"'] & D \arrow[r, "g"] & E \\
\operatorname{im} f \arrow[ur] & \ker g \arrow[u] &
\end{tikzcd}$$

$$g \circ \operatorname{im} f \circ epi = 0 \implies g \circ \operatorname{im} f = 0$$

$\implies$ $\operatorname{im} f$ factors uniquely through $\ker g$

A sequence of arrows $C \xrightarrow{f} D \xrightarrow{g} E$ is <u>exact</u> if

- $gf = 0$
- the canonical arrow $\ker g \longrightarrow \operatorname{im} f$ is an iso

$$\Updownarrow$$

$$\ker g = \operatorname{im} f$$

$$\begin{tikzcd}
C \arrow[r, "f"] & D \arrow[r, "g"] & E \\
\operatorname{im} f \arrow[ur] \arrow[r, dashed, "iso"'] & \ker g \arrow[u] &
\end{tikzcd}$$

<u>Exercise</u>

$0 \to A \xrightarrow{f} B$ exact $\iff$ f mono

$B \xrightarrow{g} C \to 0$ exact $\iff$ g epi

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0 \quad \text{exact}$$

$\iff$ • f mono • g epi • $f = \mathrm{im}\, f = \ker g$ • $\mathrm{coker}\, f = g$

<u>Consequence</u>: $\mathcal{A}$ abelian category
(C, ∂) in $Ch(\mathcal{A})$

$\partial_n \partial_{n+1} = 0 \implies$ get canonical arrow $B_n(C) \longrightarrow Z_n(C)$

<u>nth Homology of C</u> $H_n(C) := \text{target Coker}\big(B_n(C) \longrightarrow Z_n(C)\big)$

$\Big(\mathcal{A} = R\text{-mod}: \ \mathrm{coker}\big(B_n(C) \hookrightarrow Z_n(C)\big) = Z_n(C)/B_n(C)\Big)$

$$H_n(C \xrightarrow{f} D) = \begin{tikzcd}
B_n(C) \arrow[r, "\alpha"] \arrow[d, "B_n(f)"'] & Z_n(C) \arrow[r, "\mathrm{coker}\,\alpha"] \arrow[d, "Z_n(f)"] & H_n(C) \arrow[d, dashed, "\exists!"] \\
B_n(C) \arrow[r, "\beta"'] & Z_n(D) \arrow[r, "\mathrm{coker}\,\beta"'] & H_n(D)
\end{tikzcd}$$

Comes from

$$\begin{tikzcd}
& Z_n(C) \arrow[d, "\mathrm{coker}\,\alpha"] \arrow[ddr, bend left, "\mathrm{coker}\,\beta \circ Z_n(f)"] & \\
B_n(C) \arrow[ur, "\alpha"] \arrow[r, "0"'] \arrow[drr, bend right, "0"'] & H_n(C) \arrow[dr, dashed] & \\
& & H_n(D)
\end{tikzcd}$$

<u>Exercise</u> $H_n : Ch(\mathcal{A}) \longrightarrow \mathcal{A}$ is an additive functor

$\mathcal{A}$ abelian category
f, g maps of complexes in $Ch(\mathcal{A})$

A homotopy between f and g is a sequence of arrows

$F_n \xrightarrow{h_n} G_{n+1}$ such that $h_{n-1}\delta_n + \delta_{n+1}h_n = f_n - g_n$

<u>Exercise</u> Homotopic maps induce the same map in homology

<u>theorem</u> $\mathcal{A}$ abelian category, x object in $\mathcal{A}$.

$$\begin{aligned} \mathcal{A} &\longrightarrow Ab \\ y &\longmapsto Hom_{\mathcal{A}}(x,y) \end{aligned} \qquad \text{and} \qquad \begin{aligned} \mathcal{A} &\longrightarrow Ab \\ y &\longmapsto Hom_{\mathcal{A}}(y,x) \end{aligned}$$

are left exact functors

<u>Proof</u> Enough to show: $Hom_{\mathcal{A}}(x,-)$ left exact
because $Hom_{\mathcal{A}}(-,x) = Hom_{\mathcal{A}^{op}}(x,-)$

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \quad \text{exact in } \mathcal{A}$$

WTS: $0 \longrightarrow Hom_{\mathcal{A}}(x,A) \xrightarrow{f_*} Hom_{\mathcal{A}}(x,B) \xrightarrow{g_*} Hom_{\mathcal{A}}(x,C)$ exact
in Ab

- f mono $\Longrightarrow$ $f_*(-)$ is injective

- $gf = 0 \Longrightarrow g_* f_* = (gf)_* = 0_* = 0$

• $\ker g_* = \operatorname{im} f_*$ in Ab:

$h \in \mathrm{Hom}_{\mathcal{A}}(x, C)$ $\qquad gh = g_*(h) = 0$

f mono $\Rightarrow$ $f = \operatorname{im} f \underset{\substack{\uparrow \\ \text{exactness}}}{=} \ker g$

$gh = 0 \Rightarrow h$ factors uniquely through $\ker g = f$

$\Rightarrow h \in \operatorname{im} f_*$.

$\square$

Exercise $\mathcal{A}$ abelian $\Longrightarrow$ $\mathcal{A}^I$ abelian

<u>Yoneda Embedding for abelian categories</u>

$\mathcal{A}$ locally small category

$\mathcal{A} \longrightarrow \mathrm{Ab}^{\mathcal{A}^{op}}$:= Contravariant functors $\mathcal{A} \to \mathrm{Ab}$

$x \longmapsto \mathrm{Hom}_{\mathcal{A}}(-, x)$

is an embedding into a full subcategory, and it reflects exactness:

if $\mathrm{Hom}_{\mathcal{A}}(-, x) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(-, y) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(-, z)$ is exact,

then $\qquad x \longrightarrow y \longrightarrow z$ is exact

Proof • Injective on objects because Homs all disjoint

$$\begin{aligned} \mathrm{Nat}(\mathrm{Hom}_{Ab}(-,x),\ \mathrm{Hom}_{Ab}(-,y)) &\longrightarrow \mathrm{Hom}_{Ab}(x,y) \\ \eta &\longmapsto \eta_x(1_x) \end{aligned}$$

is a natural bijection by the old Yoneda lemma

this says our functor is full and faithful!

Reflects exactness:

$$\mathrm{Hom}_{Ab}(-,x) \xrightarrow{f_*} \mathrm{Hom}_{Ab}(-,y) \xrightarrow{g_*} \mathrm{Hom}_{Ab}(-,z) \quad \text{exact}$$

$$\Longrightarrow \quad gf = \underbrace{g_* f_*}_{0}(1_x) = 0$$

Also need: $\ker g = \mathrm{im}\, f$

$$g_*(\ker g) = g \circ \ker g = 0 \Longrightarrow \ker g \overset{\text{in Ab}}{\in} \ker g_* = \mathrm{im}\, f_*$$

$\Rightarrow$ $\ker g$ factors uniquely through f.

$\Rightarrow$ $\ker g$ factors through $\mathrm{im}\, f$, say by φ

$$\begin{tikzcd} x \arrow[r, "f"] & y \arrow[r, "g"] & z \\ \mathrm{im} f \arrow[ur] \arrow[r, "\psi"] & \ker g \arrow[u] \arrow[l, bend left, "\varphi"] & \end{tikzcd}$$

using universal properties of kernels/cokernels, show ψ and φ are inverses

<u>Corollary</u> $\mathcal{A} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{B}$ adjoint pair (L, R) of additive functors between abelian categories

<u>thm</u> (Freyd - Mitchell Embedding theorem)
$\mathcal{A}$ small abelian category
there exist a (possibly non-commutative) ring R and an exact, fully faithfull embedding

$$\mathcal{A} \longrightarrow R\text{-mod}$$

the snake lemma and LES in homology hold in any abelian category

<u>Towards derived functors</u> $\mathcal{A}$ abelian category

- An object P is projective if $\mathrm{Hom}_{\mathcal{A}}(P, -)$ is exact $\iff$

$$\begin{tikzcd} & P \arrow[dl, dashed] \arrow[d] \\ X \arrow[r, "\text{epi}"'] & Y \end{tikzcd}$$

- An object E is injective if $\mathrm{Hom}_{\mathcal{A}}(-, E)$ is exact $\iff$

$$\begin{tikzcd} E & \\ X \arrow[u] \arrow[r, "\text{mono}"'] & Y \arrow[ul, dashed] \end{tikzcd}$$

- A Projective resolution for M is a complex

$$P = \cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow 0 \quad \text{such that}$$

$H_0(P) = 0$, $H_n(P) = 0$ for all $n \neq 0$, and P_n is projective for all n

- An injective resolution for M is a (co)complex

$E = \quad 0 \to E^0 \to E^1 \to \cdots$ such that

$H^n(E) = 0$ for all $n \neq 0$, $H^0(E) = M$, and E_n is injective for all n.

- $\mathcal{A}$ has enough projectives if for every object M there exists an epi $I \to M$ with I projective
- $\mathcal{A}$ has enough injectives if for every object M there exists a mono $M \to E$ with E injective

Example R-mod has enough projectives and enough injectives

theorem If $\mathcal{A}$ has enough projectives, every object has a projective resolution
If $\mathcal{A}$ has enough injectives, every object has an injective resolution

Proof

$$\begin{tikzcd}
0 \arrow[dr] & & 0 & & & & \\
& \ker\partial_1 \arrow[ur] \arrow[dr] & & & & & \\
I_2 \arrow[ur, "\varepsilon_2"] \arrow[rr, dashed, "\partial_2"'] & & I_1 \arrow[rr, dashed, "\partial_1"] \arrow[dr, "\varepsilon_1"'] & & I_0 \arrow[r, "\varepsilon_0"] & M \arrow[r] & 0 \\
& & & \ker\partial_0 \arrow[ur] \arrow[dr] & & & \\
& & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

Injectives : dual

At each step, $\partial_n = \ker\partial_{n-1} \circ \varepsilon_n$, ε_n epi

$$\operatorname{im}\partial_n = \operatorname{im}(\ker\partial_{n-1} \circ \varepsilon_n) = \operatorname{im}(\underbrace{\ker\partial_{n-1}}_{\text{mono}}) = \ker\partial_{n-1}$$

■

<u>theorem</u> $\mathcal{A}$ abelian category

(I, ∂) in $Ch_{\geq 0}(\mathcal{A})$ with I_i projective for all i

(Q, δ) projective resolution for N

$I_0 \xrightarrow{\varepsilon} M$ arrow in $\mathcal{A}$ with $\varepsilon \partial_1 = 0$, $M \xrightarrow{f} N$

there exists a map of complexes $I \xrightarrow{\varphi} Q$ such that

$$\begin{array}{ccc} I_0 & \xrightarrow{\varepsilon} & M \\ \varphi_0 \downarrow & & \downarrow f \\ Q_0 & \xrightarrow[\delta_0]{} & N \end{array}$$

commutes

which is unique up to homotopy

<u>Horseshoe lemma</u> $\mathcal{A}$ abelian category

I projective resolution of A

R projective resolution of C

$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ ses in $\mathcal{A}$

there exists a projective resolution Q of B and liftings F and G of f and g (respectively) such that

$0 \to I \xrightarrow{F} Q \xrightarrow{G} R \to 0$ is a ses in $Ch(\mathcal{A})$

$$\begin{array}{ccccccccc} & & \vdots & & \vdots & & \vdots & & \\ & & \downarrow & & & & \downarrow & & \\ & & I_1 & \xrightarrow{F_1} & Q_1 & \xrightarrow{G_1} & R_1 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & I_0 & \xrightarrow[F_0]{} & Q_0 & \xrightarrow[G_0]{} & R_0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & A & \xrightarrow[f]{} & B & \xrightarrow[g]{} & C & \to & 0 \end{array}$$

Proof

Notation: $\oplus$ denotes the product $\equiv$ coproduct in $\mathcal{A}$

$f \oplus g :=$ unique arrow

$$\begin{tikzcd} & x\oplus y \arrow[d, dashed, "f\oplus g"] & \\ x \arrow[ur, "i_x"] \arrow[r, "f"'] & z & y \arrow[ul, "i_y"'] \arrow[l, "g"] \end{tikzcd}$$

$Q_n := P_n \oplus R_n$ projective

$F_n : P_n \to Q_n$, $G_n : Q_n \to R_n$ canonical arrows

Step 0

$$\begin{tikzcd} & P_0 \arrow[d, "\partial_0"'] & & R_0 \arrow[d, "\partial_0"] \arrow[dl, dashed, "\gamma"'] & \\ 0 \arrow[r] & A \arrow[r, "f"'] & B \arrow[r, "g"'] & C \arrow[r] & 0 \end{tikzcd}$$

projective

Set $\partial_0^Q := (f\partial_0) \oplus \gamma$

Universal property of the coproduct $\Longrightarrow$

$$\begin{tikzcd} & P_0 \arrow[r, "F_0"] \arrow[d, "\partial_0"'] & Q_0 \arrow[r, "G_0"] \arrow[d, "\partial_0^Q"'] & R_0 \arrow[d, "\partial_0"] \arrow[dl, dashed, "\gamma"'] & \\ 0 \arrow[r] & A \arrow[r, "f"'] & B \arrow[r, "g"'] & C \arrow[r] & 0 \end{tikzcd}$$

Commutes

Five lemma $\Longrightarrow$ ∂_0^Q epi

Snake lemma $\Longrightarrow$ ses $0 \to \ker \partial_n^P \to \ker \partial_n^Q \to \ker \partial_n^R \to 0$

Induction Repeat the base case to

$$\begin{tikzcd} 0 \arrow[r] & P_{n+1} \arrow[r, "F_{n+1}"] \arrow[d, "\partial_{n+1}^P"'] & Q_{n+1} \arrow[r, "G_{n+1}"] & R_{n+1} \arrow[r] \arrow[d, "\partial_{n+1}^R"] & 0 \\ 0 \arrow[r] & \ker \partial_n^P \arrow[r] & \ker \partial_n^Q \arrow[r] & \ker \partial_n^R \arrow[r] & 0 \end{tikzcd}$$

$\blacksquare$

Dual $\mathcal{A}$ enough injectives, $0 \to A \to B \to C \to 0$ exact

$A \to E_A$, $C \to E_C$ injective resolutions

$\Longrightarrow$ there exists an injective resolution of B and a ses

$$0 \to E_A \to E_B \to E_C \to 0$$