

Previously, on Homological Algebra:

key Example
 $R\text{-mod}$

\mathcal{A}_0 is an abelian category if:

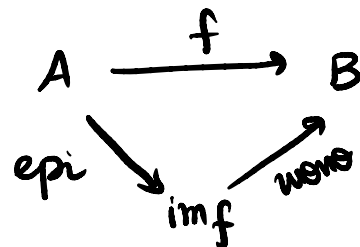
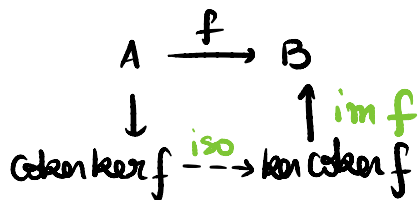
- \mathcal{A}_0 is additive ($\text{Hom}_{\mathcal{A}_0}$ are all abelian groups, \circ bilinear, \mathcal{A}_0 has a zero object and all finite products \equiv coproducts)
- \mathcal{A}_0 has all kernels and cokernels
- mono \Rightarrow kernels and epi \Rightarrow cokernel

• the canonical arrow $A \xrightarrow{f} B$ is an iso

\downarrow \uparrow
 $\text{coker ker } f \xrightarrow{\text{iso}} \text{ker coker } f$

Every arrow f factors as $f = \text{mono} \circ \text{epi}$

Image $\text{im } f = \text{ker coker } f$



Given an abelian category \mathcal{A}_0 :

- $\text{Ch}(\mathcal{A}_0)$ is an abelian category
- $\mathcal{A}_0^{\text{op}}$ is abelian

(C, ∂) in $\text{Ch}(A_0)$

$$Z_n(C) := \overset{\text{object}}{\ker \partial_n}$$

cycles

$$B_n(C) := \overset{\text{object}}{\text{im } \partial_n}$$

boundaries

Remark A_0 abelian category $C \xrightarrow{f} D \xrightarrow{g} E$ such that $gf=0$

Claim there is a canonical arrow $\text{im } f \rightarrow \ker g$

$$\begin{array}{ccccc} C & \xrightarrow{f} & D & \xrightarrow{g} & E \\ \text{epi} \searrow & & \nearrow & & \\ & \text{im } f & & \ker g & \end{array}$$

$$g \circ \text{im } f \circ \text{epi} = 0 \Rightarrow g \circ \text{im } f = 0$$

$\Rightarrow \text{im } f$ factors uniquely through $\ker g$

A sequence of arrows $C \xrightarrow{f} D \xrightarrow{g} E$ is exact if

$\left. \begin{array}{l} \bullet gf=0 \end{array} \right\}$

$\left. \begin{array}{l} \bullet \text{the canonical arrow } \ker g \rightarrow \text{im } f \text{ is an iso} \end{array} \right\}$

$$\ker g \overset{\cong}{=} \text{im } f$$

$$\begin{array}{ccccc} C & \xrightarrow{f} & D & \xrightarrow{g} & E \\ & \nearrow & & \nwarrow & \\ \text{im } f & \overset{\text{iso}}{\dashrightarrow} & \ker g & & \end{array}$$

Exercise $0 \rightarrow A \xrightarrow{f} B$ exact $\Leftrightarrow f$ mono
 $B \xrightarrow{g} C \rightarrow 0$ exact $\Leftrightarrow g$ epi

$0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ exact

\Leftrightarrow • f mono • g epi • $f = \text{im } f = \text{ker } g$ • $\text{coker } f = g$

Consequence: \mathcal{A} abelian category
 (C, ∂) in $\text{Ch}(\mathcal{A})$

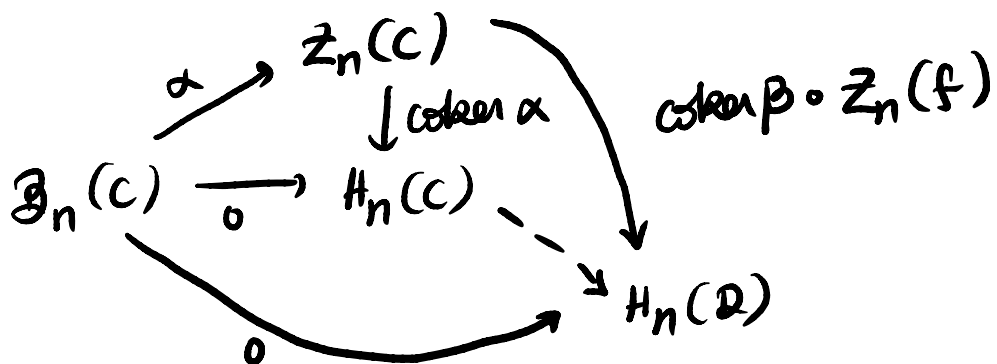
$\partial_n \partial_{n+1} = 0 \Rightarrow$ get canonical arrow $\mathcal{B}_n(C) \rightarrow \mathcal{Z}_n(C)$

n th Homology $\mathcal{H}_n(C) := \text{target Coker}(\mathcal{B}_n(C) \rightarrow \mathcal{Z}_n(C))$

($\mathcal{A} = R\text{-mod}$: $\text{coker}(\mathcal{B}_n(C) \rightarrow \mathcal{Z}_n(C)) = \mathcal{Z}_n(C) / \mathcal{B}_n(C)$)

$$H_n(C \xrightarrow{f} D) = \begin{array}{ccccc} \mathcal{B}_n(C) & \xrightarrow{\alpha} & \mathcal{Z}_n(C) & \xrightarrow{\text{coker } \alpha} & H_n(C) \\ \mathcal{B}_n(f) \downarrow & & \downarrow \mathcal{Z}_n(f) & & \downarrow \exists! \\ \mathcal{B}_n(C) & \xrightarrow{\beta} & \mathcal{Z}_n(D) & \xrightarrow{\text{coker } \beta} & H_n(D) \end{array}$$

Comes from



Exercise $H_n: \text{Ch}(\mathcal{A}) \rightarrow \mathcal{A}$ is an additive functor

\mathcal{A} abelian category

f, g maps of complexes in $\text{Ch}(\mathcal{A})$

A homotopy between f and g is a sequence of arrows

$$F_n \xrightarrow{h_n} G_{n+1} \quad \text{such that} \quad h_{n-1} \delta_n + \delta_{n+1} h_n = f_n - g_n$$

Exercise Homotopic maps induce the same map in homology

Theorem \mathcal{A} abelian category, x object in \mathcal{A} .

$$\mathcal{A} \rightarrow \text{Ab}$$

and

$$\mathcal{A} \rightarrow \text{Ab}$$

$$y \mapsto \text{Hom}_{\mathcal{A}}(x, y)$$

$$y \mapsto \text{Hom}_{\mathcal{A}}(y, x)$$

are left exact functors

Proof Enough to show: $\text{Hom}_{\mathcal{A}}(x, -)$ left exact

$$\text{because } \text{Hom}_{\mathcal{A}}(-, x) = \text{Hom}_{\mathcal{A}^{\text{op}}}(x, -)$$

$$0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0 \quad \text{exact in } \mathcal{A}$$

$$\text{WTS: } 0 \rightarrow \text{Hom}_{\mathcal{A}}(x, A) \xrightarrow{f_*} \text{Hom}_{\mathcal{A}}(x, B) \xrightarrow{g_*} \text{Hom}_{\mathcal{A}}(x, C) \text{ exact in Ab}$$

$$\bullet f \text{ mono} \Rightarrow f_*(-) \text{ is injective}$$

$$\bullet gf = 0 \Rightarrow g_* f_* = (gf)_* = 0_* = 0$$

• $\ker g_* = \text{im } f_*$ in Ab :

$$h \in \text{Hom}_{\text{Ab}}(x, C) \quad g_* h = g_*(h) = 0$$

$$f \text{ mono} \Rightarrow f = \text{im } f \underset{\substack{\uparrow \\ \text{exactness}}}{=} \ker g$$

$$g_* h = 0 \Rightarrow h \text{ factors uniquely through } \ker g = f \\ \Rightarrow h \in \text{im } f_*$$

□

Exercise \mathcal{A} abelian $\Rightarrow \mathcal{A}^{\text{I}}$ abelian

Yoneda Embedding for abelian categories

\mathcal{A} locally small category

$$\mathcal{A} \longrightarrow \text{Ab}^{\mathcal{A}^{\text{op}}} := \text{Contravariant functors } \mathcal{A} \rightarrow \text{Ab} \\ x \longmapsto \text{Hom}_{\mathcal{A}}(-, x)$$

is an embedding into a full subcategory, and it reflects exactness:

if $\text{Hom}_{\mathcal{A}}(-, x) \rightarrow \text{Hom}_{\mathcal{A}}(-, y) \rightarrow \text{Hom}_{\mathcal{A}}(-, z)$ is exact,

then $x \rightarrow y \rightarrow z$ is exact

Proof • Injective on objects because Hom's all disjoint

$$\text{Nat}(\text{Hom}_{\mathcal{A}b}(-, x), \text{Hom}_{\mathcal{A}b}(-, y)) \longrightarrow \text{Hom}_{\mathcal{A}b}(x, y)$$

$$\downarrow \qquad \qquad \qquad \downarrow$$

$$\qquad \qquad \qquad \qquad \qquad \qquad \mathbb{Z}_x(\perp_x)$$

is a natural bijection by the old Yoneda lemma

this says our functor is full and faithful!

Reflects exactness:

$$\text{Hom}_{\mathcal{A}b}(-, x) \xrightarrow{f_*} \text{Hom}_{\mathcal{A}b}(-, y) \xrightarrow{g_*} \text{Hom}_{\mathcal{A}b}(-, z) \text{ exact}$$

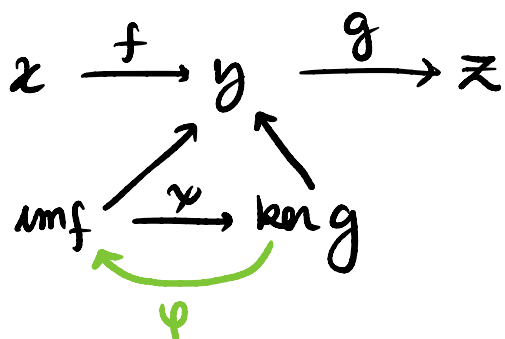
$$\Rightarrow gf = \underbrace{g_* f_*}_{0}(\perp_x) = 0$$

Also need: $\ker g = \text{im } f$

$$g_*(\ker g) = g \circ \ker g = 0 \Rightarrow \ker g \in \ker g_* = \text{im } f_*$$

$\Rightarrow \ker g$ factors uniquely through f .

$\Rightarrow \ker g$ factors through $\text{im } f$, say by ψ



using universal properties of kernels/cokernels, show ψ and φ are inverses

Corollary $\mathcal{A} \overset{L}{\underset{R}{\rightleftarrows}} \mathcal{B}$ adjoint pair (L, R) of additive functors between abelian categories

thm (Freyd-Mitchell Embedding theorem)
 \mathcal{A} small abelian category

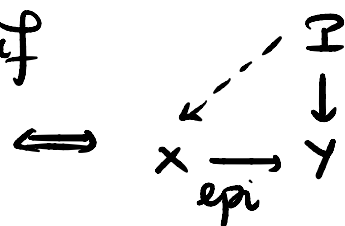
there exist a (possibly non-commutative) ring R
 and an exact, fully faithful embedding
 $\mathcal{A} \rightarrow R\text{-mod}$

the Snake lemma and LES in homology hold in any abelian category

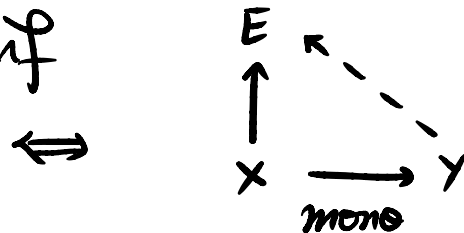
Towards derived functors

\mathcal{A} abelian category

• An object P is projective if
 $\text{Hom}_{\mathcal{A}}(P, -)$ is exact



• An object E is injective if
 $\text{Hom}_{\mathcal{A}}(-, E)$ is exact



• A Projective resolution for M is a complex

$$P = \cdots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow 0 \quad \text{such that}$$

$H_0(P) = M, H_n(P) = 0$ for all $n \neq 0$, and P_n is projective for all n

• An injective resolution for M is a (co)Complex

$$E = 0 \rightarrow E^0 \rightarrow E^1 \rightarrow \dots \text{ such that}$$

$$H^n(E) = 0 \text{ for all } n \neq 0, H^0(E) = M, \text{ and } E_n \text{ is injective for all } n.$$

• \mathcal{A} has enough projectives if for every object M there exists an epi $I \rightarrow M$ with I projective

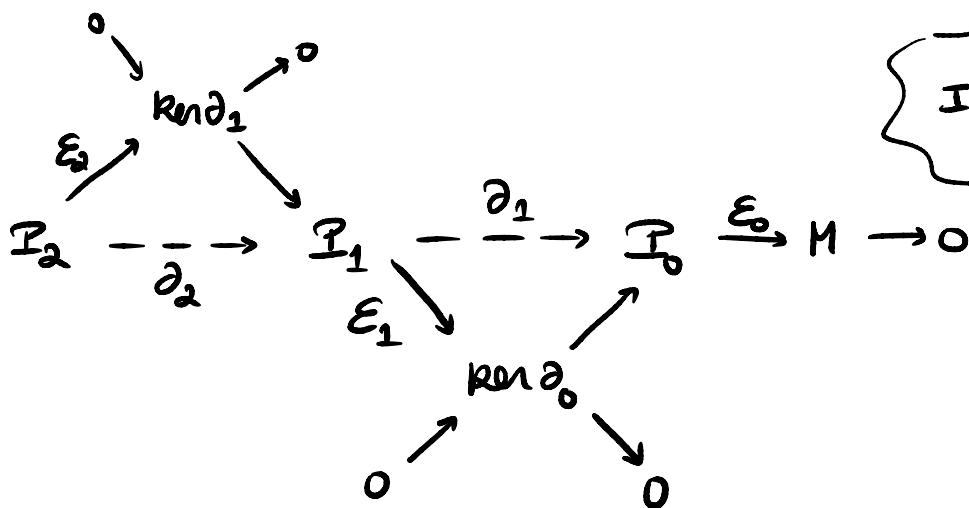
• \mathcal{A} has enough injectives if for every object M there exists a mono $M \rightarrow E$ with E injective

Example R -mod has enough projectives and enough injectives

Theorem If \mathcal{A} has enough projectives, every object has a projective resolution

If \mathcal{A} has enough injectives, every object has an injective resolution

Proof



Injectives: dual

At each step, $\partial_n = \ker \partial_{n-1} \circ \epsilon_n$, ϵ_n epi

$$\text{im } \partial_n = \text{im} (\ker \partial_{n-1} \circ \epsilon_n) = \text{im} (\underbrace{\ker \partial_{n-1}}_{\text{mono}}) = \ker \partial_{n-1}$$

□

Theorem of abelian category

(\mathcal{I}, ∂) in $\text{Ch}_{\geq 0}(\mathcal{A})$ with \mathcal{I}_i projective for all i

(\mathcal{Q}, δ) projective resolution for N

$\mathcal{I}_0 \xrightarrow{\varepsilon} M$ arrow in \mathcal{A} with $\varepsilon \partial_1 = 0$, $M \xrightarrow{f} N$

there exists a map of complexes $\mathcal{I} \xrightarrow{\varphi} \mathcal{Q}$ such that

$$\begin{array}{ccc} \mathcal{I}_0 & \xrightarrow{\varepsilon} & M \\ \varphi_0 \downarrow & & \downarrow f \\ \mathcal{Q}_0 & \xrightarrow{\delta_0} & N \end{array} \quad \text{commutes}$$

which is unique up to homotopy

Horseshoe lemma of abelian category

\mathcal{I} projective resolution of A

\mathcal{R} projective resolution of C

$$0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0 \quad \text{ses in } \mathcal{A}$$

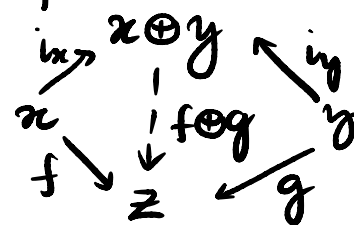
there exists a projective resolution \mathcal{Q} of B and liftings F and G of f and g (respectively) such that

$$0 \rightarrow \mathcal{I} \xrightarrow{F} \mathcal{Q} \xrightarrow{G} \mathcal{R} \rightarrow 0 \quad \text{is a ses in } \text{Ch}(\mathcal{A})$$

$$\begin{array}{ccccccc} & & \vdots & & \vdots & & \\ & & \downarrow & & \downarrow & & \\ & & \mathcal{I}_1 & \xrightarrow{F_1} & \mathcal{Q}_1 & \xrightarrow{G_1} & \mathcal{R}_1 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & \mathcal{I}_0 & \xrightarrow{F_0} & \mathcal{Q}_0 & \xrightarrow{G_0} & \mathcal{R}_0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C \rightarrow 0 \end{array}$$

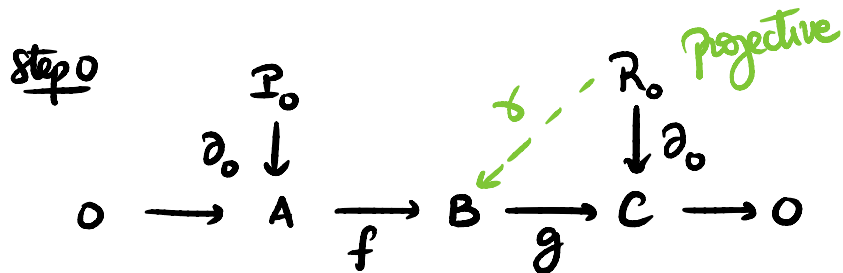
Proof Notation: \oplus denotes the product \equiv coproduct in \mathcal{A}

$f \oplus g :=$ unique arrow



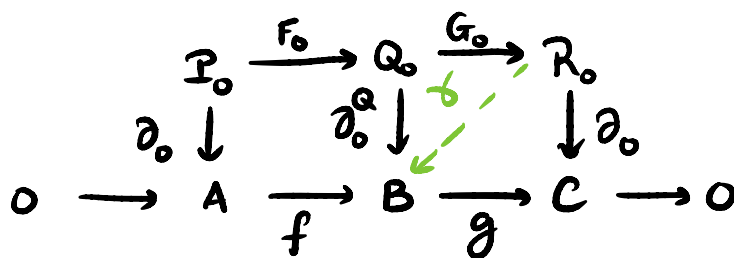
$Q_n := P_n \oplus R_n$ projective

$F_n: P_n \rightarrow Q_n, G_n: R_n \rightarrow Q_n$ canonical arrows



Set $d_0^Q := (f d_0) \oplus \sigma$

Universal property
of the coproduct \Rightarrow

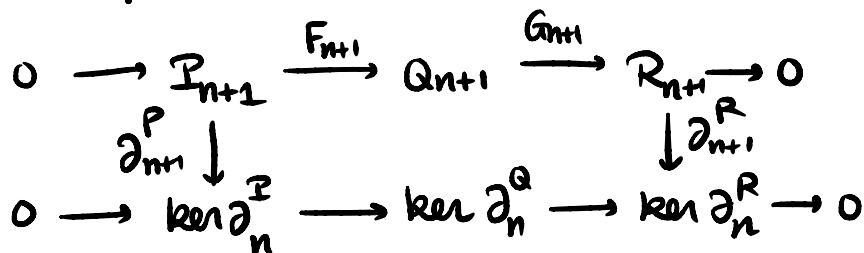


Commutates

Five lemma $\Rightarrow d_0^Q$ epi

Snake lemma \Rightarrow ses $0 \rightarrow \ker d_n^P \rightarrow \ker d_n^Q \rightarrow \ker d_n^R \rightarrow 0$

Induction Repeat the base case to



□

Dual \mathcal{A} enough injectives, $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ exact

$A \rightarrow E_A, C \rightarrow E_C$ injective resolutions

\Rightarrow there exists an injective resolution of B and a ses

$$0 \rightarrow E_A \rightarrow E_B \rightarrow E_C \rightarrow 0$$