Previously, on Homological Algebra:

$\mathcal{A}$ abelian category

- An object P is projective if $\mathrm{Hom}_{\mathcal{A}}(P,-)$ is exact $\iff$

$$\begin{tikzcd} & P \arrow[dl, dashed] \arrow[d] \\ X \arrow[r, "\text{epi}"'] & Y \end{tikzcd}$$

- An object E is injective if $\mathrm{Hom}_{\mathcal{A}}(-,E)$ is exact $\iff$

$$\begin{tikzcd} E & \\ X \arrow[u] \arrow[r, "\text{mono}"'] & Y \arrow[ul, dashed] \end{tikzcd}$$

- A Projective resolution for M is a complex

$$P = \cdots \to P_2 \to P_1 \to P_0 \to 0 \quad \text{such that}$$

$H_0(P) = 0$, $H_n(P) = 0$ for all $n \neq 0$, and P_n is projective for all n

- An injective resolution for M is a (co)complex

$$E = \quad 0 \to E^0 \to E^1 \to \cdots \quad \text{such that}$$

$H^n(E) = 0$ for all $n \neq 0$, $H^0(E) = M$, and E_n is injective for all n.

- $\mathcal{A}$ has enough projectives if for every object M there exists an epi $P \to M$ with P projective

- $\mathcal{A}$ has enough injectives if for every object M there exists a mono $M \to E$ with E injective

Example R-mod has enough projectives and enough injectives things we will need today:

- F additive functor $\mathcal{A} \longrightarrow \mathcal{B}$

  $\Longrightarrow$ get a functor $F: Ch(\mathcal{A}) \longrightarrow Ch(\mathcal{B})$

- Additive functors preserve homotopies

  so if $\varphi \simeq \psi \Longrightarrow F(\varphi) \simeq F(\psi)$

- A ses
$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"'] & B \arrow[r, "g"'] \arrow[l, dashed, bend right, "q"'] & C \arrow[r] \arrow[l, dashed, bend right, "r"'] & 0 \end{tikzcd}$$
splits if

  - $\exists q: B \to A$ with $qf = id_A$
  - $\exists r: C \to B$ with $gr = id_B$
  - $\cong$ $0 \to A \xrightarrow{i} A \oplus C \xrightarrow{\pi} C \to 0$

- A injective or C projective $\Longrightarrow$ $0 \to A \to B \to C \to 0$ splits

- Additive functors preserve $\underline{\underline{\text{split}}}$ short exact sequences

$$0 \longrightarrow A \to B \to C \to 0 \quad \text{split ses}$$
$$\Downarrow$$
$$0 \to FA \to FB \to FC \to 0 \quad \text{split ses}$$

Goal: for a ring R, get long Exact sequences for $\mathrm{Hom}_R(M,-)$, $\mathrm{Hom}_R(-,M)$, and $M \otimes_R -$

- $\mathrm{Hom}_R(M,-)$ is left exact:

$$0 \to A \to B \to C \to 0 \text{ ses}$$
$$\Downarrow$$
$$0 \to \mathrm{Hom}_R(M,A) \to \mathrm{Hom}_R(M,B) \to \mathrm{Hom}_R(M,C) \to ?$$

- $\mathrm{Hom}_R(-,M)$ is left exact:

$$0 \to A \to B \to C \to 0 \text{ ses}$$
$$\Downarrow$$
$$0 \to \mathrm{Hom}_R(M,C) \to \mathrm{Hom}_R(M,B) \to \mathrm{Hom}_R(M,A) \to ?$$

- $M \otimes_R -$ is right exact:

$$0 \to A \to B \to C \to 0 \text{ ses}$$
$$\Downarrow$$
$$? \to M \otimes_R A \to M \otimes_R B \to M \otimes_R C \to 0$$

Solution derived functors.

P projective	E injective	P projective
$\Downarrow$	$\Downarrow$	$\Downarrow$
$\mathrm{Hom}_R(P,-)$ exact	$\mathrm{Hom}_R(-,E)$ exact	$P \otimes_R -$ exact

Derived functors of $\mathcal{A} \xrightarrow{F} \mathcal{B}$ $\quad$ $\mathcal{A}, \mathcal{B}$ abelian

- F right exact covariant functor

If $\mathcal{A}$ has enough projectives, the left derived functors of F are

$$L_i F : \mathcal{A} \longrightarrow \mathcal{B} \qquad i \geq 0$$

$$\begin{cases} \bullet\ L_i F(A) := H^i(F(P)) & P \xrightarrow{\sim} A \text{ projective resolution} \\ \bullet\ L_i F(f) := H^i(F(\varphi)) & A \xrightarrow{f} B \quad P \xrightarrow{\varphi} Q \text{ lift of } f \\ & Q \xrightarrow{\sim} B \text{ projective resolution} \end{cases}$$

- F left exact covariant functor

If $\mathcal{A}$ has enough injectives, the right derived functors of F are

$$R^i F : \mathcal{A} \longrightarrow \mathcal{B} \qquad i \geq 0$$

$$\begin{cases} \bullet\ R^i F(A) := H^i(F(E)) & A \xrightarrow{\sim} E \text{ injective resolution} \\ \bullet\ R^i F(f) := H^i(F(\varphi)) & A \xrightarrow{f} B \quad E \xrightarrow{\varphi} I \text{ lift of } f \\ & B \xrightarrow{\sim} I \text{ injective resolution} \end{cases}$$

- F contravariant left exact functor

If $\mathcal{A}$ has enough projectives, the right derived functors of F are

$$R^i F : \mathcal{A} \longrightarrow \mathcal{B} \qquad i \geq 0$$

$$\begin{cases} \bullet\ R^i F(A) := H^i(F(P)) & P \xrightarrow{\sim} A \text{ projective resolution} \\ \bullet\ R^i F(f) := H^i(F(\varphi)) & A \xrightarrow{f} B \quad P \xrightarrow{\varphi} Q \text{ lift of } f \\ & Q \xrightarrow{\sim} B \text{ projective resolution} \end{cases}$$

- F contravariant right exact functor

If $\mathcal{A}$ has enough injectives, the left derived functors of F are

$$L_iF : \mathcal{A} \longrightarrow \mathcal{B} \qquad i \geq 0$$

$$\begin{cases} \bullet\ L_iF(A) := H_i(F(E)) \\ \bullet\ L_iF(f) := H_i(F(\varphi)) \end{cases}$$

$A \xrightarrow{\sim} E$ injective resolution

$A \xrightarrow{f} B \qquad E \xrightarrow{\varphi} I$ lift of f

$B \xrightarrow{\sim} I$ injective resolution

original functor is	Exactness	Use	take	derived functor
Covariant	Left	injectives	H^i	Covariant
	Right	projectives	H_i	covariant
Contravariant	Left	projectives	H^i	Contravariant
	Right	injectives	H_i	contravariant

Notes:

- F exact $\Rightarrow H_i(F(C)) = F(H_i(C)) \Rightarrow \begin{cases} L_iF = 0 \\ R^iF = 0 \end{cases}$ for all $i > 0$
- P projective $\Rightarrow \underbrace{0 \to P \to 0}_{\text{projective resolution}} \Rightarrow L_iF(P) = 0$ for $i > 0$
- E injective $\Rightarrow \underbrace{0 \to E \to 0}_{\text{injective resolution}} \Rightarrow R^iF(E) = 0$ for $i > 0$

<u>Prop</u>: $\mathcal{A}$ abelian category with enough projectives

F covariant right exact functor

① $L_i F(A)$ well-defined up to iso for all objects A

② $L_i F(f)$ well-defined for every arrow $A \xrightarrow{f} B$

③ $L_i F$ additive functor

④ $L_0 F = F$ (naturally isomorphic)

<u>Proof</u> ① P, Q projective resolutions for A

$\exists$ maps of complexes $P \xrightarrow{\varphi} Q \xrightarrow{\chi} P$

Such that $\varphi\chi \simeq 1_Q \quad \chi\varphi \simeq 1_P$

Additive functors preserve homotopies

$\Rightarrow F(\varphi)F(\chi) \simeq 1_{F(Q)}, \quad F(\chi)F(\varphi) \simeq 1_{F(P)}$

Homotopic maps induce the same map in homology

$\Rightarrow F(\varphi)$ and $F(\chi)$ induce isos in homology

$\Rightarrow \quad H_i(F(P)) \underset{\text{iso}}{\cong} H_i(F(Q))$

② Fix projective resolutions $P \xrightarrow{\simeq} A$ and $Q \xrightarrow{\simeq} B$

Any two lifts φ, ψ of f are homotopic

$$\begin{tikzcd} P \arrow[r] \arrow[d, dashed, shift right, "\varphi"'] \arrow[d, dashed, shift left, "\psi"] & A \arrow[d, "f"] \\ Q \arrow[r] & B \end{tikzcd}$$

- Additive functors preserve homotopies

$\Rightarrow F(\varphi), F(\psi)$ are homotopic

- Homotopic maps induce the same map in homology

$\Rightarrow H_i(F(\varphi)) = H_i(F(\psi))$ ✓

③ Idea: $L_i F = H_i \cdot F$ is a composition of additive functors

④ $P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0$ exact

F right exact $\Longrightarrow$ $F(P_1) \longrightarrow F(P_0) \longrightarrow F(A) \longrightarrow 0$ exact

$\Rightarrow H_0(F(P)) = \operatorname{coker}(F(P_1) \longrightarrow F(P_0)) = F(A)$

$\Bigg($ $\ker(F(P_0) \longrightarrow 0) = 1_{F(P_0)} \Rightarrow \operatorname{im} F(\partial_1) \xrightarrow{\text{canonical}} F(P_0) = \operatorname{im} F(\partial_1)$

exactness $\Rightarrow \operatorname{im} F(\partial_1) = \ker(F(P_0) \xrightarrow{\text{epi}} F(A))$

epi = coker ker $\Rightarrow F(P_0) \longrightarrow F(A) = \operatorname{coker} \operatorname{im} F(\partial_1)$ $\Bigg)$

Remark $\mathcal{A}$ has enough injectives $\Rightarrow \mathcal{A}^{op}$ has enough projectives

F (covariant) left exact $\Rightarrow$ F^{op} right exact

$$R^i F = \left(L_i(F^{op})\right)^{op}$$

theorem $\mathcal{A}$ abelian category with enough projectives

F right exact covariant functor

Any ses $0 \to A \to B \to C \to 0$ gives rise to a LES

$$\cdots \to L_2F(C) \to L_1F(A) \to L_1F(B) \to L_1F(C) \to F(A) \to F(B) \to F(C) \to 0$$

Proof Fix projective resolutions

$$P \xrightarrow{\sim} A \qquad R \xrightarrow{\sim} C$$

Horseshoe lemma $\Rightarrow$ Can construct resolution Q of B st

$$0 \to P \to Q \to R \to 0 \quad \text{ses}$$

$\Rightarrow$ For each n, $0 \to P_n \to Q_n \to R_n \to 0$ <u>split</u> ses

$\Rightarrow$ $0 \to F(P_n) \to F(Q_n) \to F(R_n) \to 0$ ses for all n

$\Rightarrow$ $0 \to F(P) \to F(Q) \to F(R) \to 0$ ses

the LES in homology is the sequence we are looking for

Back to R-mod: Ext and Tor

R ring

$$\mathrm{Tor}_i^R(M,N) := L_i(M\otimes_R -)(N) \overset{\text{iso}}{=} L_i(-\otimes_R N)(M)$$

$$\mathrm{Ext}_R^i(M,N) := R^i\,\mathrm{Hom}_R(M,-)(N) \overset{\text{iso}}{=} R^i\,\mathrm{Hom}_R(-,N)(M)$$

Double complex family of objects $\{C_{p,q}\}_{p,q\in\mathbb{Z}}$ together with arrows $d^h : C_{p,q} \longrightarrow C_{p-1,q}$ and $d^v : C_{p,q} \longrightarrow C_{p,q-1}$ satisfying $d^h d^h = 0$, $d^v d^v = 0$, $d^h d^v + d^v d^h = 0$

$$\begin{tikzcd}
 & \vdots \arrow[d, "d^v"'] & \vdots \arrow[d] & \\
\cdots & C_{p-1,q+1} \arrow[l] \arrow[d, "d^v"'] & C_{p,q+1} \arrow[l, "d^h"'] \arrow[d] & \cdots \arrow[l] \\
\cdots & C_{p-1,q} \arrow[l] \arrow[d, "d^v"'] & C_{p,q} \arrow[l, "d^h"'] \arrow[d] & \cdots \arrow[l] \\
 & \vdots & \vdots &
\end{tikzcd}$$

the total complex of a double complex C is the complex $\mathrm{Tot}^{\oplus}(C)$

$$\mathrm{Tot}^{\oplus}(C)_n := \bigoplus_{p+q=n} C_{p,q}$$

with differential $d := d^h + d^v$

the Product total complex of C is $\mathrm{Tot}^{\pi}(C)$ with

$$\mathrm{Tot}^{\pi}(C)_n = \prod_{p+q=n} C_{p,q} \quad \text{and differential } d = d^h + d^v$$

<u>Most important examples</u> R-mod

① tensor product double complex of C and D

$$(C \otimes D)_{p,q} = C_p \otimes_R D_q$$

$$d^h := \partial^C \otimes_R 1_D \qquad d^v := (-1)^p \, 1_C \otimes_R \partial^D$$

tensor product total complex = tensor product of C and D

$$(C \otimes D)_n := \bigoplus_{p+q=n} C_p \otimes_R D_q$$

$$d(x \otimes y) = \partial(x) \otimes y + (-1)^p x \otimes \partial(y)$$

$$x \in C_p, \; y \in D_q$$

② Hom double complex of C and D in $Ch(R)$

$$\mathrm{Hom}(C, D)_{p,q} = \mathrm{Hom}_R(C_{-p}, D_q)$$

with differentials $\mathrm{Hom}_R(C_{-p}, D_q) \longrightarrow \mathrm{Hom}_R(C_{-p-1}, D_q)$

$$f \longmapsto f \circ \partial^C$$

and $\mathrm{Hom}_R(C_{-p}, D_q) \longrightarrow \mathrm{Hom}_R(C_{-p}, D_{q-1})$

$$f \longmapsto (-1)^{p+q+1} \partial^D \circ f$$

Internal Hom complex of C and D $= \mathrm{Tot}^{\pi}(\mathrm{Hom}(C,D))$

$$\mathrm{Hom}(C,D)_n = \prod_{p+q=n} \mathrm{Hom}_R(C_{-p}, D_q)$$

with differential $d(f) = f \circ \partial^C + (-1)^{p+q+1} \partial^D \circ f$

Remarks

1) $Z_0(\mathrm{Hom}(C,D)) \equiv \mathrm{Hom}_{Ch(R)}(C,D)$

because:

0-cycle $\Leftrightarrow$ $C_k \xrightarrow{f_k} D_k \in \mathrm{Hom}_R(C_k, D_k)$ such that

$$f \circ \partial^C - \partial^D \circ f = d(f) = 0$$

2) $B_0(\mathrm{Hom}(C,D)) \equiv$ homotopies of maps $C \to D$

because

0-boundary $\Leftrightarrow C_k \xrightarrow{f_k} D_k \in \mathrm{Hom}_R(C_k, D_k)$ such that

$\exists \quad C_k \xrightarrow{h_k} D_{k+1} \in \mathrm{Hom}_R(C_k, D_{k+1})$

with $d(h) = f \Leftrightarrow f_k = \partial^D \circ h_k - h_{k-1} \circ \partial^C$