Previously on Homological Algebra:

- Derived functors : definition
- derived functors give rise to LES from every ses

theorem $\mathcal{A}$ abelian category with enough projectives

F right exact covariant functor

Any ses $0 \to A \to B \to C \to 0$ gives rise to a LES

$$\cdots \to L_2F(C) \to L_1F(A) \to L_1F(B) \to L_1F(C) \to F(A) \to F(B) \to F(C) \to 0$$

theorem $\mathcal{A}$, $\mathcal{B}$ abelian categories, $\mathcal{A}$ has enough projectives

F right exact covariant functor

$T_i : \mathcal{A} \to \mathcal{B}$ additive covariant functors

Assume:

- Every ses $0 \to A \to B \to C \to 0$ in $\mathcal{A}$ gives rise to a LES

$$\cdots \to T_2(C) \to T_1(A) \to T_1(B) \to T_1(C) \to T_0(A) \to T_0(B) \to T_0(C) \to 0$$

- T_0 is naturally isomorphic to F
- $T_n(P) = 0$ for all projectives P, $n \geqslant 1$.

then T_i is naturally isomorphic to L_iF.

Acyclic Assembly Lemma C double complex in R-mod.

If either

1) C upper half plane double complex with exact rows

2) C right half plane double complex with exact columns

then $\mathrm{Tot}^{\oplus}(C)$ is exact

If either

3) C upper half plane double complex with exact columns

4) C right half plane double complex with exact rows

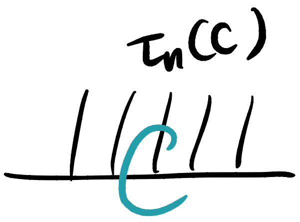
then $\mathrm{Tot}^{\pi}(C)$ is exact

Proof 1) $\Leftrightarrow$ 2) by switching indexes

3) $\Leftrightarrow$ 4) by switching indexes

Claim 3) $\Rightarrow$ 2)

$$\tau_n(C)_{p,q} := \begin{cases} C_{p,q} & q > n \\ \ker(C_{p,n} \xrightarrow{d^v} C_{p,n-1}) & q = n \\ 0 & q < n \end{cases}$$



there is an obvious map $\tau_n(C) \longrightarrow C$, iso in homology $\geq n$

Given C right half plane double complex with exact rows

$\Rightarrow \tau_n(C)$ is an upper half plane double complex with exact columns

$\overset{3)}{\Rightarrow}$ $\mathrm{Tot}^{\pi}(\tau_n(C))$ exact //////

<u>Note</u> only finitely many nonzero $C_{p,q}$ for each fixed $n=p+q$

$\Rightarrow \mathrm{Tot}^{\pi}(\tau_n(C)) = \mathrm{Tot}^{\oplus}(\tau_n(C))$ exact

Now we claim this implies $\mathrm{Tot}^{\oplus}(C)$ is exact.

- If $a = (a_{p,q}) \in Z_k(\mathrm{Tot}^{\oplus}(C))$, only finitely many $a_{p,q} \neq 0$

Now pick $n < \min\{q \mid a_{p,q} \neq 0\}$

then $a \in Z_k(\mathrm{tot}^{\oplus}(\tau_n(C))) = B_k(\mathrm{Tot}^{\oplus}(\tau_n(C))) \subseteq B_k(\mathrm{Tot}^{\oplus}(C))$

$\Rightarrow H_k(\mathrm{Tot}^{\oplus}(C)) = c$

Now just have to show 3):

C upper half plane double complex with exact columns

Want to show: $\mathrm{Tot}^{\pi}(C)$ is exact //////

Will show: $H_0(\mathrm{Tot}^{\pi}(C)) = 0$, all others follow by shifting

$c \in Z_0(\mathrm{Tot}^{\pi}(C)) \Leftrightarrow c = (c_p)$, $c_p \in C_{-p,p}$ for each $p \geq 0$

$d(c) = 0 \Leftrightarrow d^v(c_p) + d^h(c_{p-1}) = 0$
for all p

$$\begin{tikzcd} c_p \arrow[d] & \\ \star & c_{p-1} \arrow[l, mapsto] \end{tikzcd}$$

Will construct $b = (b_{-p,p+1})$ $\quad b_{-p,p+1} \in C_{-p,p+1}$

$$d(b) = c \Leftrightarrow d^v(b_{-p,p+1}) + d^h(b_{-p+1,p}) = c_p \quad \text{for all } p$$

$\rightsquigarrow$ so $c \in B_0(\mathrm{Tot}^\pi(C))$

Induction:

step 0:

Set $b_{1,0} = 0 \in C_{-1,0}$ $\quad (p=-1)$

Notice $\overset{v}{d}(c_{0,0}) = 0$ $\quad$ ($\downarrow$ exact!)

$\Rightarrow \exists\, b_{0,1} \in C_{0,1} : \overset{v}{d}(b_{0,1}) = c_{0,0}$ $\quad$ so:

$$d^v(b_{0,1}) + d^h(b_{1,0}) = c_{0,0} \checkmark$$

$$\begin{tikzcd}
C_{0,1} \arrow[d] & \\
c_{0,0} \arrow[d, "d^v"] & b_{1,0} \in C_{1,0} \arrow[l] \\
C_{0,-1} = 0 &
\end{tikzcd}$$

step n: Suppose we have $b_{-s+1,s}$, $-1 \le s \le p$

$$d^v(b_{-s,s+1}) + d^h(b_{-s+1,s}) = c_s$$

Now:

$$d^v(c_{-p,p} - d^h(b_{-p+1,p})) \overset{\substack{d^vd^h + d^hd^v = 0 \\ \downarrow}}{=} d^v(c_p) + d^hd^v(b_{-p+1,p})$$

$d^v(b_{-p+1,p}) + d^h(b_{-p+2,p-1}) = c_{p-1}$

induction hypothesis

$$= d^v(c_p) + d^h(c_{p-1} - d^h(b_{-p+2,p+1}))$$

$$= d^v(c_p) + d^h(c_{p-1}) - \underbrace{d^hd^h}_{=0}(b_{-p+2,p+1})$$

$$= d^v(c_p) + d^h(c_{p-1})$$

$$= (d(c))_{p-1} = 0$$

$$\therefore\ d^v(c_{-p,p} - d^h(b_{-p+1,p})) = 0$$

$$\Longrightarrow\ c_{-p,p} - d^h(b_{-p+1,p}) \in \ker d^v \overset{\text{exact columns!}}{=} \operatorname{im} d^v$$

$$\Longrightarrow\ \exists\ b_{-p,p+1} : \quad d^v(b_{-p,p+1}) = c_{-p,p} - d^h(b_{-p+1,p})$$

$$\therefore\ c_{-p,p+1} = d^v(b_{-p,p+1}) + d^h(b_{-p+1,p})$$ ☺

□

Notation kth suspension of the complex C is

$$\Sigma^k C := \text{complex } D \text{ with } (\Sigma^k C)_n = C_{n-k} = C[k]$$
$$\partial^{\Sigma^k C} = (-1)^k \partial^C$$

Cone $C \xrightarrow{f} D$ map of complexes

the cone of f is the complex $\operatorname{cone}(f)$ with

$$\operatorname{cone}(f)_n := C_{n-1} \oplus D_n$$

$$\partial_n^{\operatorname{cone}(f)} := \begin{pmatrix} -\partial^C & 0 \\ -f & \partial^D \end{pmatrix} :$$

$$\begin{tikzcd} C_{n-1} \arrow[r, "-\partial^C"] \arrow[ddr, "-f"] & C_{n-2} \\ \oplus & \oplus \\ D_n \arrow[r, "\partial_D"'] & D_{n-1} \end{tikzcd}$$

Exercise to give a map of complexes $\operatorname{cone}(f) \longrightarrow E$ is to give

- a map of complexes $D \xrightarrow{g} E$
- a nullhomotopy for gf

Any map of complexes $C \xrightarrow{f} D$ gives rise to a ses

$$0 \to D \to \text{cone}(f) \to \Sigma C \to 0$$

determined by the canonical arrows in the product $\equiv$ coproduct

the connecting homomorphisms from the Snake lemma

$$H_{n-1}(C) = H_n(\Sigma C) \xrightarrow{\delta} H_{n-1}(\mathcal{D})$$

are exactly $H_{n-1}(C) \xrightarrow{H_{n-1}(f)} H_{n-1}(D)$

$\Longrightarrow$ f quasi-iso $\Longleftrightarrow$ cone(f) exact.

Any map of complexes $C \xrightarrow{f} D$ gives rise to a double complex:

$$X = \begin{tikzcd}
 & \vdots \arrow[d, "\partial"'] & \vdots \arrow[d, "-\partial"] & \\
0 & \mathcal{D}_2 \arrow[l] \arrow[d, "\partial"'] & C_2 \arrow[l, "f"'] \arrow[d, "-\partial"] & 0 \arrow[l] \\
0 & \mathcal{D}_1 \arrow[l] \arrow[d, "\partial"'] & C_1 \arrow[l, "f"] \arrow[d] & 0 \arrow[l] \\
 & \vdots & \vdots &
\end{tikzcd}$$

$$\text{Tot}^{\oplus}(X) = \text{cone}(f)$$

Balancing Tor R ring

M, N R-modules

$P \xrightarrow{\sim} M$, $Q \xrightarrow{\sim} N$ projective resolutions

For every n,

$$L_n(M \otimes_R -)(N) = H_n(M \otimes_R Q) \cong H_n(P \otimes_R N) = L_n(- \otimes_R N)(M)$$

Proof

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
 & P_0 \otimes Q_2 \arrow[d] & P_1 \otimes Q_2 \arrow[l] & P_2 \otimes Q_2 \arrow[l] \arrow[d] & \cdots \arrow[l] \\
P \otimes Q = & P_0 \otimes Q_1 \arrow[d] & P_1 \otimes Q_1 \arrow[l] \arrow[d] & P_2 \otimes Q_1 \arrow[l] \arrow[d] & \cdots \arrow[l] \\
 & P_0 \otimes Q_0 & P_1 \otimes Q_0 \arrow[l] & P_2 \otimes Q_0 \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

P_i, Q_i projective $\Rightarrow$ flat $\Longrightarrow$ our rows and colums are exact

except for the 0^{th} row/column

exact rows

$$\begin{tikzcd}
 & & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
 & M \otimes Q_2 \arrow[d] & P_0 \otimes Q_2 \arrow[l] \arrow[d] & P_1 \otimes Q_2 \arrow[l] & P_2 \otimes Q_2 \arrow[l] \arrow[d] & \cdots \arrow[l] \\
R = & M \otimes Q_1 \arrow[d] & P_0 \otimes Q_1 \arrow[l] \arrow[d] & P_1 \otimes Q_1 \arrow[l] \arrow[d] & P_2 \otimes Q_1 \arrow[l] \arrow[d] & \cdots \arrow[l] \\
 & M \otimes Q_0 & P_0 \otimes Q_0 \arrow[l] & P_1 \otimes Q_0 \arrow[l] & P_2 \otimes Q_0 \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

exact columns

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
 & P_0 \otimes Q_2 \arrow[d] & P_1 \otimes Q_2 \arrow[l] & P_2 \otimes Q_2 \arrow[l] \arrow[d] & \cdots \arrow[l] \\
C = & P_0 \otimes Q_1 \arrow[d] & P_1 \otimes Q_1 \arrow[l] \arrow[d] & P_2 \otimes Q_1 \arrow[l] \arrow[d] & \cdots \arrow[l] \\
 & P_0 \otimes Q_0 \arrow[d] & P_1 \otimes Q_0 \arrow[l] \arrow[d] & P_2 \otimes Q_0 \arrow[l] & \cdots \arrow[l] \\
 & P_0 \otimes M & P_1 \otimes M \arrow[l] & P_2 \otimes M \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

Acyclic Assembly Lemma $\Longrightarrow$ $\begin{matrix} \mathrm{Tot}^{\oplus}(C) \\ \mathrm{Tot}^{\oplus}(D) \end{matrix}$ are exact

Exercise

$$\mathrm{Cone}\left(\mathrm{Tot}^{\oplus}(P \otimes Q) \xrightarrow{P \otimes \varepsilon} P \otimes_R N\right) = \Sigma\, \mathrm{Tot}^{\oplus}(R)$$

$$\mathrm{cone}\left(\mathrm{Tot}^{\oplus}(P \otimes Q) \xrightarrow{\pi \otimes Q} M \otimes_R Q\right) = \Sigma\, \mathrm{Tot}^{\oplus}(C)$$

then $\mathrm{Cone}(P \otimes \varepsilon)$ and $\mathrm{Cone}(\pi \otimes Q)$ are exact

$\Rightarrow$ $P \otimes \varepsilon$, $\pi \otimes Q$ are quasi-isos

so $H_i(P \otimes_R N) \cong H_i(\mathrm{Tot}^{\oplus}(P \oplus Q)) \cong H_i(M \oplus Q)$

$\blacksquare$

Balancing Ext R ring

M, N R-modules

$P \xrightarrow[\pi]{\sim} M$ projective resolution

$N \xrightarrow[\varepsilon]{\sim} E$ injective resolution

For every n,

$$R^n \mathrm{Hom}_R(M,-)(N) = H^n(\mathrm{Hom}_R(M,E)) \cong H^n(\mathrm{Hom}_R(P,N)) = R^n \mathrm{Hom}_R(-,N)(M)$$

Proof Same idea, with

$$\begin{tikzcd}
\mathrm{Hom}_R(M, E^2) \arrow[r] & \mathrm{Hom}_R(P_0, E^2) \arrow[r] & \mathrm{Hom}_R(P_1, E^2) \arrow[r] & \mathrm{Hom}_R(P_2, E^2) \\
\mathrm{Hom}_R(M, E^1) \arrow[r] \arrow[u] & \mathrm{Hom}_R(P_0, E^1) \arrow[r] \arrow[u] & \mathrm{Hom}_R(P_1, E^1) \arrow[r] \arrow[u] & \mathrm{Hom}_R(P_2, E^1) \arrow[u] \\
\mathrm{Hom}_R(M, E^0) \arrow[r] \arrow[u] & \mathrm{Hom}_R(P_0, E^0) \arrow[r] \arrow[u] & \mathrm{Hom}_R(P_1, E^0) \arrow[r] \arrow[u] & \mathrm{Hom}_R(P_2, E^0) \arrow[u] \\
 & \mathrm{Hom}_R(P_0, N) \arrow[r] \arrow[u] & \mathrm{Hom}_R(P_1, N) \arrow[r] \arrow[u] & \mathrm{Hom}_R(P_2, N) \arrow[u]
\end{tikzcd}$$

theorem Every ses of R-modules

$$0 \to A \to B \to C \to 0$$

induces long exact sequences

$$\cdots \to \mathrm{Tor}^R_{n+1}(M,C) \to \mathrm{Tor}^R_n(M,A) \to \mathrm{Tor}^R_n(M,B) \to \mathrm{Tor}^R_n(M,C) \to \cdots$$

$$\cdots \to \mathrm{Tor}^R_1(M,C) \to A\otimes_R M \to B\otimes_R M \to C\otimes_R M \to 0$$

and

$$0 \to \mathrm{Hom}_R(M,A) \to \mathrm{Hom}_R(M,B) \to \mathrm{Hom}_R(M,C) \to \mathrm{Ext}^1_R(M,A) \to \cdots$$

$$\cdots \to \mathrm{Ext}^{n-1}_R(M,C) \to \mathrm{Ext}^n_R(M,A) \to \mathrm{Ext}^n_R(M,B) \to \cdots$$

and

$$0 \to \mathrm{Hom}_R(C,M) \to \mathrm{Hom}_R(B,M) \to \mathrm{Hom}_R(A,M) \to \mathrm{Ext}^1_R(C,M) \to \cdots$$

$$\cdots \to \mathrm{Ext}^{n-1}_R(A,M) \to \mathrm{Ext}^n_R(C,M) \to \mathrm{Ext}^n_R(B,M) \to \cdots$$

<u>theorem</u> R ring

M, N R-modules

$\mathrm{Tor}^R_i(M,N) \cong \mathrm{Tor}^R_i(N,M)$ for all i

<u>Example</u> $n>1$ integer, M $\mathbb{Z}$-module

Let's compute $\mathrm{Tor}^{\mathbb{Z}}_i(\mathbb{Z}/n, M)$ and $\mathrm{Ext}^i_{\mathbb{Z}}(\mathbb{Z}/n, M)$

$$0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/n \to 0$$

is a projective resolution for $\mathbb{Z}/n$.

$$\mathrm{Tor}_i^{\mathbb{Z}}(\mathbb{Z}/n, M) = H_i\left(0 \to M \otimes_{\mathbb{Z}} \mathbb{Z} \xrightarrow{1 \otimes n} M \otimes_{\mathbb{Z}} \mathbb{Z} \to 0\right)$$

$$\begin{tikzcd}
0 \arrow[r] & M \otimes_{\mathbb{Z}} \mathbb{Z} \arrow[r, "1 \otimes n"] \arrow[d, phantom, "\cong" rotate=90] & M \otimes_{\mathbb{Z}} \mathbb{Z} \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & 0 \\
0 \arrow[r] & M \arrow[r, "n"'] & M \arrow[r] & 0
\end{tikzcd}$$

$m \otimes k \longmapsto mk$

so: $\ker(M \xrightarrow{\cdot n} M) = (0 :_M n)$

$\mathrm{coker}(M \xrightarrow{\cdot n} M) = M/nM$

$$\mathrm{Tor}_i^{\mathbb{Z}}(\mathbb{Z}/n, M) = \begin{cases} 0, & i \geq 2 \\ (0 :_M n), & i = 1 \\ M/nM, & i = 0 \end{cases}$$

$$\mathrm{Ext}^i_{\mathbb{Z}}(\mathbb{Z}/n, M) = H^i\left(0 \to \underset{0}{\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, M)} \xrightarrow{n^*} \underset{1}{\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, M)} \to 0\right)$$

$\mathrm{Hom}_{\mathbb{Z}}(-, M)$ contravariant!

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, M) \arrow[r, "n"] \arrow[d, "\cong"] & \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, M) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & M \arrow[r, "n"'] & M \arrow[r] & 0
\end{tikzcd}$$

$f \longmapsto f(1)$

$nf \longmapsto n \cdot f(1)$

$0 \qquad\qquad 1$

$$\mathrm{Ext}^{i}_{\mathbb{Z}}(\mathbb{Z}/n, M) = \begin{cases} 0 & i \neq 1, 0 \\ (0 :_{M} n) & i = 0 \\ M/nM & i = 1 \end{cases}$$