Previously, on Homological Algebra:

Derived functors of $\mathcal{F}$: $L_i F / R^i F / R_i F / L^i F$

(Co)homology of $\mathcal{F}$(injective/projective resolution of $-$)

using $\downarrow$ derived functors

Given ses $0 \to A \to B \to C \to 0$, we get a LES

the left/right derived functors of $\mathcal{F}$ are the only functors satisfying certain properties:

theorem $\mathcal{F}, T_i : \mathcal{A} \to \mathcal{B}$ left covariant functor between abelian categories

$T_i : \mathcal{A} \to \mathcal{B}$ covariant functors satisfying:

1. $T_0 \cong \mathcal{F}$ (naturally isomorphic)
2. Given a ses $0 \to A \to B \to C \to 0$, there is a LES
$0 \to T_0(A) \to T_0(B) \to T_0(C) \to T_1(A) \to T_1(B) \to \cdots$
3. $T_i(E) = 0$ for all injectives E

then $T_i \cong L_i F$ for all i (naturally isomorphic)

Main Examples: Ext and Tor

$$\mathrm{Tor}_i^R(M,N) := H_i\big((\underbrace{P \xrightarrow{\sim} M}_{\text{projective resolution}}) \otimes_R N\big) = H_i\big(M \otimes_R (\underbrace{Q \xrightarrow{\sim} N}_{\text{projective resolution}})\big)$$

$$\mathrm{Ext}_R^i(M,N) := H^i\big(\mathrm{Hom}_R(M, \underbrace{N \xrightarrow{\sim} E}_{\text{injective resolution}})\big) = H^i\big(\mathrm{Hom}_R(\underbrace{P \xrightarrow{\sim} M}_{\text{projective resolution}}, N)\big)$$

Last time: $n \in \mathbb{Z}$, M any abelian group

$$\mathrm{Tor}_i^{\mathbb{Z}}(\mathbb{Z}/n, M) = \begin{cases} M/nM & i=0 \\ (0 :_M n) & i=1 \\ 0 & i \geq 2 \end{cases}$$

$$\mathrm{Ext}^i_{\mathbb{Z}}(\mathbb{Z}/n, M) = \begin{cases} (0 :_M n) & i=0 \\ M/nM & i=1 \\ 0 & i \geq 2 \end{cases}$$

We can also find these from a LES:

$$0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}/n \to 0 \quad \text{ses}$$

$$\rightsquigarrow \mathrm{Hom}_R(-, M)$$

LES:

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, M) \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, M) \xrightarrow{n^*} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, M) \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n, M) \to 0$$

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, M) \cong \mathbb{Z} \xrightarrow{n} \mathbb{Z} \cong \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, M)$$

$0 = \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}, M)$ ($\mathbb{Z}$ projective)

so $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n, M) = \mathrm{coker}(\mathbb{Z} \xrightarrow{n} \mathbb{Z}) = M/nM$.

Easy facts:

- R Noetherian, M, N fg $\Rightarrow$ $\mathrm{Ext}^i_R(M, N)$, $\mathrm{Tor}_i^R(M, N)$ fg
- $\mathrm{ann}_R(M)$, $\mathrm{ann}_R(N)$ kill $\mathrm{Ext}^i_R(M, N)$, $\mathrm{Tor}_i^R(M, N)$

R ring, M R-module

the projective dimension of M is

$$\mathrm{pdim}_R(M) := \inf\left\{ c \;\middle|\; \begin{array}{l} 0 \to P_c \to \cdots \to P_0 \to 0 \\ \text{is a projective resolution of } M \end{array} \right\}$$

the injective dimension of M is

$$\mathrm{injdim}_R(M) := \inf\left\{ c \;\middle|\; \begin{array}{l} 0 \to E_0 \to \cdots \to E_c \to 0 \\ \text{is an injective resolution of } M \end{array} \right\}$$

In the graded/local setting,

projdim (M) := length of a minimal free resolution

<u>Note</u> $\mathrm{pdim}(M) = 0 \iff M$ projective

$\mathrm{injdim}(M) = 0 \iff M$ injective

Projective/injective dimension can be infinite

<u>Example</u> $R = k[x]/(x^2)$ (local ring), $M = R/(x) = k = R/\mathfrak{m}$

$$\cdots \longrightarrow R \xrightarrow{\cdot x} R \xrightarrow{\cdot x} R \xrightarrow{\cdot x} R \longrightarrow k \longrightarrow 0$$

$\mathrm{pdim}_R(k) = \infty$

Remark If $\mathrm{pdim}_R(M) = n < \infty$, then

$\mathrm{Tor}_i^R(M,-) = 0$ and $\mathrm{Ext}_R^i(M,-) = 0$ for $i > n$

Also, $\mathrm{injdim}_R(M) = n < \infty \implies \mathrm{Ext}_R^i(-,M) = 0$ for $i > n$

theorem

(R, m, k) Noetherian local ring or $\mathbb{N}$-graded k-algebra, $m = R_+$, $R_0 = k$

If M is a fg (graded) R-module,

$$\beta_i(M) = \dim_k \mathrm{Tor}_i^R(M,k) = \dim_k \mathrm{Ext}_R^i(M,k)$$

Proof F minimal free resolution for M

$$\cdots \to F_i \xrightarrow{\varphi_i} F_{i-1} \to \cdots \to F_0 \to 0$$

F minimal $\implies$ all the entries in φ_i are in m

$$F_i \otimes_R k = R^{\beta_i(M)} \otimes_R k = k^{\beta_i(M)}$$

so $\varphi_i \otimes_R k =$ take all the entries of φ_i as elements of $k \overset{\text{all in } m!}{=} 0$

so $F \otimes_R k := \cdots \xrightarrow{0} k^{\beta_i(M)} \xrightarrow{0} \cdots \xrightarrow{0} k^{\beta_0(M)} \to 0$

$\implies \mathrm{Tor}_i^R(M,k) = k^{\beta_i(M)} \implies \beta_i(M) = \dim_k \mathrm{Tor}_i^R(M,k)$

$\mathrm{Hom}_R(F_i, k) = \mathrm{Hom}_R(R^{\beta_i(M)}, k) \cong k^{\beta_i(M)}$

$\mathrm{Hom}_R(\varphi_i, k)$ = transpose φ_i, see all entries in k $= 0$

$\mathrm{Hom}_R(F, k) = 0 \to k^{\beta_0(M)} \xrightarrow{0} k^{\beta_1(M)} \xrightarrow{0} \cdots \xrightarrow{0} k^{\beta_i(M)} \to \cdots$

$\mathrm{Ext}^i_R(M, k) = k^{\beta_i(M)} \Rightarrow \beta_i(M) = \dim_k \mathrm{Ext}^i_R(M, k)$ □

Corollary $\mathrm{pdim}_R(M) \leq \mathrm{pdim}_R(R)$

Exercise $\beta_{i,j}(M) = \dim_k (\mathrm{Tor}_i^R(M,N))_j = \dim_k (\mathrm{Ext}^i_R(M,N))_j$

Hopes and dreams Can we an explicit minimal free resolution of k find?

Koszul complex R ring, M R-module

- on $x \in R$ is the complex

$$K(x) := 0 \to R \xrightarrow{\cdot x} R \to 0$$

- on $\underline{x} = x_1, \dots, x_n$ is defined by

$$K(\underline{x}) = K(x_1, \dots, x_n) := K(x_1, \dots, x_{n-1}) \otimes_R K(x_n)$$

- on M with respect to $\underline{x} = x_1, \dots, x_n$ is

$$K(\underline{x}; M) = K(x_1, \dots, x_n; M) := K(x_1, \dots, x_n) \otimes_R M$$

Ex

$$K(x,y) = \begin{array}{c} 0 \to R \xrightarrow{\cdot x} R \to 0 \\ \otimes \\ 0 \to R \xrightarrow[\cdot y]{} R \to 0 \end{array}$$

⇝ double complex

$$\begin{array}{ccc} R\otimes R & \xleftarrow{x} & R\otimes R \\ y \downarrow & & \downarrow -y \\ R\otimes R & \xleftarrow[x]{} & R\otimes R \end{array}$$

(diagonals labelled 0, 1, 2; columns labelled 0, 1)

⇒ total complex

$$0 \longrightarrow R \xrightarrow{\begin{pmatrix} -y \\ x \end{pmatrix}} R^2 \xrightarrow{(x \;\; y)} R \longrightarrow 0$$

Koszul complex on x, y

<u>Alternative</u> <u>construction</u>: using exterior algebras

$$\Lambda M = R \oplus M \oplus M \otimes M \oplus M \otimes M \otimes M \oplus \cdots \Big/ \langle x\otimes y + y \otimes x ,\, x \otimes x \rangle$$

($M = \Lambda^1 M$, $M\otimes M \to \Lambda^2 M$, $M\otimes M\otimes M \to \Lambda^3 M$)

with product denoted $\wedge$ is a skew commutative algebra:

$$a \wedge b = (-1)^{\deg a \deg b}\, b \wedge a \qquad \text{for } a, b \text{ homogeneous}$$

$\Lambda^i M :=$ piece of degree i $\big(\underbrace{M \otimes \cdots \otimes M}_{i \text{ times}}\big)$

so R^n free on basis $e_1, \ldots, e_n \Rightarrow \Lambda^i R^n = R^{\binom{n}{k}}$ with basis

$$e_{i_1} \wedge \cdots \wedge e_{i_s}$$
$$1 \le i_1 < \cdots < i_s \le n$$

$$k(x_1, \dots, x_n) := 0 \to \Lambda^n R^n \longrightarrow \Lambda^{n-1} R^n \to \cdots \to \Lambda^1 R^n \to R \to 0$$
$$= 0 \to R \to R^n \to \cdots \to \underset{i}{R^{\binom{n}{i}}} \to \cdots \to R^n \to R \to 0$$

with differential

$$d(e_{i_1} \wedge \cdots \wedge e_{i_s}) = \sum_{j=1}^{s} (-1)^{j+1} x_{i_j} \, e_{i_1} \wedge \cdots \wedge \hat{e}_{i_j} \wedge \cdots \wedge e_{i_s}$$

(delete $\hat{e}_{i_j}$)

so the matrices have entries $\pm x_i$

and

$$k(x_1, \dots, x_n; M) := k(x_1, \dots, x_n) \otimes_R M$$

Example

$$k(x, y) = 0 \to \overset{\binom{2}{2}}{R} \xrightarrow{\begin{pmatrix} -y \\ x \end{pmatrix}} \overset{\binom{2}{1}}{R^2} \xrightarrow{(x \ y)} \overset{\binom{2}{0}}{R} \to 0$$

$$d(e_1) = x \qquad d(e_2) = y$$

$$d(e_1 \wedge e_2) = x e_2 + (-1)^1 y e_1 = x e_2 - y e_1$$

ith koszul homology of M with respect to $\underline{x} = x_1, \dots, x_n$

$$H_i(\underline{x}; M) := H_i(k(\underline{x}; M))$$

Properties : R ring

$\underline{x} = x_1, \dots, x_n \in R$

$I = (x_1, \dots, x_n)$

M R-module

① $H_i(\underline{x}; M) = 0$ for $i < 0$ or $i > n$

② $H_0(\underline{x}; M) = M / IM$

③ $H_n(\underline{x}; M) = (0 :_M I) = \mathrm{ann}_M(I)$

④ $H_i(\underline{x}; M)$ is killed by $\mathrm{ann}(M)$ for all i

⑤ $H_i(\underline{x}; M)$ is killed by I for all i

⑥ M Noetherian $\Longrightarrow$ $H_i(\underline{x}; M)$ Noetherian for all i

⑦ $H_i(\underline{x}; -)$ is a covariant additive functor

⑧ Every ses $0 \to A \to B \to C \to 0$ gives rise to a LES

$$\cdots \to H_{i+1}(\underline{x}; C) \to H_i(\underline{x}; A) \to H_i(\underline{x}; B) \to H_i(\underline{x}; C) \to \cdots$$

$$\cdots \to H_1(\underline{x}; C) \to H_0(\underline{x}; A) \to H_0(\underline{x}; B) \to H_0(\underline{x}; C) \to 0.$$

Sketch : ① $K_i(\underline{x}; M) = 0$ for $i > n$ and $i < 0$

② $H_0(\underline{x}; M) = H\left(M^n \xrightarrow{(x_1, \dots, x_n)} M \to 0\right) = M/IM$

③ $H_n(\underline{x}; M) = H\left(0 \to M \xrightarrow{\begin{pmatrix} x_1 \\ -x_2 \\ \vdots \\ \pm x_n \end{pmatrix}} M^n \to 0\right)$

$= \{ m \in M \mid x_1 m = -x_2 m = x_3 m = \cdots = \pm x_n m = 0 \}$

$= (0 :_M I)$

④ $K_i(\underline{x}; M) = M^{\binom{n}{i}}$ killed by $\operatorname{ann}(M)$

⑤ $a \in I \Rightarrow \cdot a$ is null homotopic

$a = a_1 x_1 + \cdots + a_n x_n$

Idea: $K_i(\underline{x}; M) \xrightarrow{(a_1 e_1 + \cdots + a_n e_n) \wedge -} K_{i+1}(\underline{x}; M)$

is a null homotopy of $K_i(\underline{x}; M) \xrightarrow{\cdot x} K_i(\underline{x}; M)$

⑥ M Noetherian $\Rightarrow$ M^k Noetherian, and all its submodules and quotients

⑦ $M \xrightarrow{f} N \rightsquigarrow K(\underline{x}; M) \xrightarrow{K(\underline{x}) \otimes f} K(\underline{x}; N)$

$\rightsquigarrow$ take homology

⑧ $K(\underline{x})_i$ free for all $i \Rightarrow K(x) \otimes_R -$ exact

$\Rightarrow 0 \to K(\underline{x}) \otimes_R A \to K(\underline{x}) \otimes_R B \to K(\underline{x}) \otimes_R C \to 0$ ses

$\Rightarrow$ LES in homology, where $H_{-1}(K(\underline{x}) \otimes_R A) = 0$

Remark $C = k(x_1, \dots, x_i ; M)$

$$k(x_1, \dots, x_{i+1} ; M) = C \otimes k(x_{i+1})$$

so

$$\left[k(x_1, \dots, x_{i+1} ; M)\right]_n = \underset{1}{C_{n-1} \otimes R} \oplus \underset{0}{C_n \otimes_R R} \cong C_{n-1} \oplus C_n$$

$$\begin{array}{ccccc} & \xleftarrow{d} & C_{n-1} \otimes R & & \\ & & \downarrow (-1)^{n-1} x_{i+1} & & \\ & & C_{n-1} \otimes R & \xleftarrow{d} & C_n \otimes R \end{array}$$

$$d(\underset{1}{a \otimes r} + \underset{0}{b \otimes s}) = d(a) \otimes r + (-1)^{n-1} a \otimes d(r) + d(b) \otimes s + 0$$

$$= d(a) \otimes r + (-1)^{n-1} x_{i+1} a \otimes r + d(b) \otimes s$$

so $d = \begin{pmatrix} d_C & 0 \\ (-1)^{n-1} x_{i+1} & d_C \end{pmatrix}$ and

$$k(x_1, \dots, x_{i+1} ; M) = \operatorname{Cone}\left(k(x_1, \dots, x_i ; M) \xrightarrow{\cdot x_{i+1}} k(x_1, \dots, x_i ; M) \right)$$

$\rightsquigarrow$ get LES

$$\cdots \to H_n(x_1, \dots, x_i ; M) \xrightarrow{\cdot x_{i+1}} H_n(x_1, \dots, x_i ; M) \to H_{n-1}(x_1, \dots, x_{i+1} ; M) \to \cdots$$

$$\begin{cases} \Rightarrow \ker\left(H_n(x_1, \dots, x_i ; M) \longrightarrow H_{n-1}(x_1, \dots, x_{i+1} ; M)\right) \\ = \operatorname{im}\left(H_n(x_1, \dots, x_i ; M) \xrightarrow{\cdot x_{i+1}} H_n(x_1, \dots, x_i ; M)\right) = x_{i+1} H_n(x_1, \dots, x_i ; M) \\ \Rightarrow \operatorname{im}\left(H_n(x_1, \dots, x_i ; M) \longrightarrow H_{n-1}(x_1, \dots, x_{i+1} ; M)\right) = \ker(x_{i+1}) \end{cases}$$

$$0 \to \frac{H_n(x_1, \dots, x_i ; M)}{x_{i+1} H_n(x_1, \dots, x_i ; M)} \longrightarrow H_{n-1}(x_1, \dots, x_{i+1} ; M) \longrightarrow \operatorname{ann}_{H_{n-1}(x_1, \dots, x_i ; M)}(x_{i+1}) \to 0$$

Regular sequences :

R ring, M R-module

$x \in R$ is regular on M if $xm = 0 \Rightarrow m = 0$ for any $m \in M$

$x_1, \dots, x_n \in R$ is a regular sequence on M if

- $(x_1, \dots, x_n) M \neq M$
- x_i regular on $M / (x_1, \dots, x_{i-1}) M$ for each i

Remark x_i regular on $M/(x_1, \dots, x_{i-1})M$

$$\Leftrightarrow \left((x_1, \dots, x_{i-1})M :_M x_i \right) = (x_1, \dots, x_{i-1}) M$$

$$\Leftrightarrow \operatorname{ann}_{M/(x_1, \dots, x_{i-1})M} (x_i) = 0$$

Examples

① $R = k[x_1, \dots, x_n]$, k field

$x_1, \dots, x_n$ is a regular sequence on R

② $R = k[x, y, z]$, k field

xy, yz not a regular sequence

③ $R = k[x, y, z]$, k field

$x, (x-1)y, (x-1)z$ is regular, but

$(x-1)y, (x-1)z, x$ is not a regular sequence

order matters !

Remark $x \in R$ is regular $\Leftrightarrow \ker(M \xrightarrow{\cdot x} M) = (0:_M x) = 0$

$\Leftrightarrow H_1(K(x; M)) = 0$

theorem $\underline{x} = x_1, \ldots, x_n$ regular sequence on M

$\Rightarrow H_i(\underline{x}; M) = 0$ for all $i \geq 1$

Proof Induction on n. $n=1$: Remark.

$n > 1$: Suppose $H_j(x_1, \ldots, x_i; M) = 0$ for all $j \geq 1$

LES:

$$\cdots \to \underbrace{H_j(x_1, \ldots, x_i; M)}_{\substack{=0 \\ j \geq 1}} \to H_j(x_1, \ldots, x_i; M) \to \underbrace{H_{j-1}(x_1, \ldots, x_i; M)}_{=0 \text{ for } j-1 \geq 1} \xrightarrow{x_{i+1}} \cdots$$

$\Rightarrow = 0$

$$0 \to \frac{H_j(x_1, \ldots, x_i; M)}{x_{i+1} H_j(x_1, \ldots, x_i; M)} \to H_{j-1}(x_1, \ldots, x_{i+1}; M) \to \operatorname{ann}_{H_{j-1}(x_1, \ldots, x_i; M)}(x_{i+1}) \to 0$$

x_{i+1} regular on $M/(x_1, \ldots, x_i)M = H_0(x_1, \ldots, x_i; M)$

$\Rightarrow \operatorname{ann}_{H_0(x_1, \ldots, x_i; M)}(x_{i+1}) = 0$

$$0 \to \underbrace{\frac{H_1(x_1, \ldots, x_i; M)}{x_{i+1} H_1(x_1, \ldots, x_i; M)}}_{\substack{\| \\ 0 \\ \text{by hypothesis}}} \to H_1(x_1, \ldots, x_{i+1}; M) \to \underbrace{\operatorname{ann}_{H_0(x_1, \ldots, x_i; M)}(x_{i+1})}_{=0} \to 0$$

$\Rightarrow H_1(x_1, \ldots, x_{i+1}; M) = 0$

Corollary $\underline{x} = x_1, \dots, x_n$ regular sequence on R

$\Rightarrow$ the koszul complex $K(\underline{x})$ is a free resolution for $R/(x_1, \dots, x_n)$

In the graded/local case, this free resolution is minimal

Hilbert Syzygy theorem k field

Every fg graded module over $R = k[x_1, \dots, x_d]$ has

$$\operatorname{pdim}(M) \leq d.$$

Proof the koszul complex $K(x_1, \dots, x_d)$ is a minimal free resolution of $R/(x_1, \dots, x_d) = k$ (!) so

$$\operatorname{pdim}_R(k) = d$$

$$\therefore \quad \operatorname{pdim}_R(M) \leq \operatorname{pdim}_R(k) \leq d.$$