A long time ago, in a quarter far far away...

R Noetherian ring $\iff$ every ideal in R is fg

$\iff$ every ascending sequence of ideals stops

M Noetherian R-module $\iff$ every submodule of M is fg

$\iff$ every ascending chain of submodules stops

R Noetherian ring : M Noetherian $\iff$ M fg

quotients, submodules of Noetherian modules are Noetherian

Canonical examples $\dfrac{k[x_1,\dots,x_n]}{I}$ and $\dfrac{k[[x_1,\dots,x_n]]}{I}$ are Noetherian rings

R ring

M R-module

A prime $\mathfrak{P}$ is associated to M if

- $\mathfrak{P} = \mathrm{ann}_R(m)$ for some $m \in M$

$\Updownarrow$

- $R/\mathfrak{P} \hookrightarrow M$

$\mathrm{Ass}(M) :=$ associated primes of M

Facts

- $M \neq 0 \Rightarrow \mathrm{Ass}(M) = \emptyset$
- R Noetherian, M fg $\Rightarrow$ $\mathrm{Ass}(M)$ finite
- zerodivisors on M $= \bigcup_{\mathfrak{P} \in \mathrm{Ass}(M)} \mathfrak{P}$

- $\underbrace{\mathrm{Min}(M)}_{\text{minimal primes}} \subseteq \mathrm{Ass}(M)$

P is a minimal prime of M if P is minimal over $\mathrm{ann}(M)$

the minimal primes in $\mathrm{Ass}(M)$ are precisely the minimal primes of M

Dimension

$\dim(R) := \sup\{n \mid P_0 \subsetneq \cdots \subsetneq P_n,\ P_i \text{ prime in } R\}$

$\mathrm{ht}(P) := \sup\{n \mid P_0 \subsetneq \cdots \subsetneq P_n = P,\ P_i \text{ prime in } R\}$

$\mathrm{ht}(I) := \min\{\mathrm{ht}(P) \mid P \in \underbrace{\mathrm{Min}(I)}_{\text{minimal primes containing } I}\}$

Note: (R, m) local $\quad \dim(R) = \mathrm{ht}(m)$

Krull's Height theorem $\quad \mathrm{ht}(x_1, \dots, x_n) \leq n$

Note: minimal primes of R have height 0

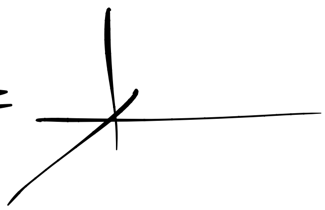
$\mathrm{ht}(x) = 1 \iff x$ not in any minimal prime of R

$\dim(k[x_1, \dots, x_d]) = d$

Idea $\quad X \subseteq \mathbb{A}^n_k$ variety $\longleftrightarrow$ ideal $I(X) \subseteq R = k[x_1, \dots, x_d]$

$\dim(X) = \dim(R/I) \qquad \operatorname{codim}(X) = \operatorname{ht}(I)$

geometric idea $\qquad$ algebraically defined

eg: $X =$  $\Longleftrightarrow I = (xy, xz, yz)$

$= (x,y) \cap (y,z) \cap (x,z)$

dim 1 $\qquad \operatorname{ht} I = 2$

codim $3 - 1 = 2 \qquad \dim(R/I) = 1$

Previously, on Homological algebra

(R, m, k) local/graded, M fg (graded) R-module

$\beta_i(M) = \dim_R(\operatorname{Tor}_i^R(M,k)) = \dim_k(\operatorname{Ext}^i_R(M,k))$

Corollary $\quad \operatorname{pdim}_R(M) \leq \operatorname{pdim}_R(k)$

Koszul complex $\quad K(x) := 0 \to R \xrightarrow{\cdot x} R \to 0$

$K(\underline{x}) = K(x_1, \dots, x_n) := K(x_1, \dots, x_{n-1}) \otimes_R K(x_n)$

$K(\underline{x}; M) = K(x_1, \dots, x_n; M) := K(x_1, \dots, x_n) \otimes_R M$

$$0 \to R \to R^n \to \cdots \to \underset{i}{R^{\binom{n}{i}}} \to \cdots \to R^n \to R \to 0$$

the matrices have entries $\pm x_i$

ith koszul homology $H_i(\underline{x}; M) := H_i(K(\underline{x}; M))$

<u>theorem</u> If $x_1, \dots, x_d$ is regular sequence on M, then $H_i(\underline{x}; M) = 0$ for $i > 0$

$\Rightarrow$ the koszul complex is a minimal free resolution for $R/(x_1, \dots, x_d)$

<u>Corollary</u> (Hilbert syzygy theorem) $R = k[x_1, \dots, x_d]$

Every fg graded R-module M has $\operatorname{pdim}_R(M) \leq d$

Proof $x_1, \dots, x_d$ is a regular sequence

$\Rightarrow$ the koszul complex on $x_1, \dots, x_d$ is a minimal free resolution for $k = R/(x_1, \dots, x_d)$

$\therefore \operatorname{pdim}_R(k) = d$

thus $\operatorname{pdim}_R(M) \leq \operatorname{pdim}_R(k) \leq d$.

<u>theorem</u> (R, m, k) Noetherian local ring or fg graded k-algebra, $R_0 = k$, $m = R_+$

$0 \neq M$ fg (graded) R-module

If $H_i(\underline{x}; M) = 0$ for all $i \geq 1$, then $\underline{x}$ is a regular sequence on M

<u>Proof</u> Induction on n

$n = 1$ is an easy remark from before.

$\underline{n>1}$:

$$0 \to \frac{H_j(x_1,\dots,x_{n-1};M)}{x_n H_j(x_1,\dots,x_{n-1};M)} \longrightarrow \underbrace{H_j(x_1,\dots,x_n;M)}_{\substack{=0 \\ \text{by assumption}}} \longrightarrow \operatorname{ann}_{H_j(x_1,\dots,x_{n-1};M)}(x_n) \to 0$$

$$\Rightarrow \quad \frac{H_j(x_1,\dots,x_{n-1};M)}{x_n H_j(x_1,\dots,x_{n-1};M)} = 0 \quad \text{and} \quad \operatorname{ann}_{H_j(x_1,\dots,x_{n-1};M)}(x_n) = 0$$

$$\Rightarrow H_j(x_1,\dots,x_{n-1};M) = x_n H_j(x_1,\dots,x_{n-1};M)$$

$$\overset{\text{NAK}}{\Rightarrow} H_j(x_1,\dots,x_{n-1};M) = 0 \quad \text{for all } j$$

$$\overset{\text{induction}}{\Rightarrow} x_1,\dots,x_{n-1} \text{ regular on } M$$

also have $\operatorname{ann}_{\underbrace{H_0(x_1,\dots,x_{n-1};M)}_{M/(x_1,\dots,x_{n-1})M}}(x_n) = 0$

<u>Corollaries</u> Local / graded case

- order doesn't matter
- $x_1,\dots,x_n$ regular $\Rightarrow$ $x_1^{a_1},\dots,x_n^{a_n}$ regular

<u>theorem</u> If $x_1, \dots, x_n$ is a regular sequence on R,

$$\mathrm{ht}(x_1, \dots, x_n) \leq n.$$

<u>Proof</u> Induction on n.

<u>$n=1$</u>: x_1 regular $\Leftrightarrow$ x_1 not a zerodivisor

$\Longleftrightarrow$ x_1 not in any associated prime

$\Rightarrow$ x_1 not in any minimal prime

$\Rightarrow$ $\mathrm{ht}(x_1) \geqslant 1$

Krull's Height theorem $\Rightarrow$ $\mathrm{ht}(x_1) = 1$

<u>$n>1$</u> x_n regular on $R/(x_1, \dots, x_{n-1})$

$$\overset{n=1}{\Longrightarrow} \mathrm{ht}\left((x_1, \dots, x_n)/(x_1, \dots, x_{n-1})\right) = 1$$

induction hypothesis $\Rightarrow$ $\mathrm{ht}(x_1, \dots, x_{n-1}) = n-1$

$\therefore$ $\mathrm{ht}(x_1, \dots, x_n) = n$. (Notice that we only get $\mathrm{ht}(x_1, \dots, x_n) \geqslant n$, but KHT finishes the job)

<u>Regular rings</u> (R, m, k) regular local ring if $\dim(R) = d$ and $\mu(m) = d$

__Idea__ Geometrically, think about varieties of dimension d that embed in $\mathbb{A}^d_k$ (so very nice and smooth)

__Localization Problem__ R regular $\overset{?}{\Longrightarrow}$ R_I regular?

Is regularity a local property?

__theorem__ (Serre) (R, m, k) Noetherian local ring $\mu(m) = s$

$$\dim_k \left(\operatorname{Tor}_i^R(k, k)\right) \geq \binom{s}{i}$$

__Idea__ F_i minimal free resolution of k

For each i, $K_i(\underline{x})$ is a direct summand of F_i

__Conjecture__ (Buchsbaum - Eisenbud, Horrocks)

(R, m, k) Noetherian local ring of dim d   open!

M Artinian R-module of finite projective dimension

$$\beta_i(M) = \dim_k \left(\operatorname{Tor}_i^R(M, k)\right) \geq \binom{d}{i}$$

there's lots of evidence for the BEH. the strongest evidence is

__theorem__ (Walker, 2017) __Total Rank Conjecture__

R Noetherian local ring, char $\neq 2$

$M \neq 0$ fg R-module of $\operatorname{pdim}_R(M) < \infty$, $c = \operatorname{ht}(\operatorname{ann} M)$

$$\sum_i \beta_i(M) \geq 2^c$$

Theorem (Auslander – Buchsbaum, Serre)

(R, m, k) Noetherian local ring of dimension d.

TFAE: ① $\operatorname{pdim}_R(k) < \infty$

② $\operatorname{pdim}_R(M) < \infty$ for all fg R-module M

③ m is generated by a regular sequence

④ m is generated by d elements

Proof

② $\Rightarrow$ ① Take $M = k$

① $\Rightarrow$ ② because $\operatorname{pdim}_R(M) \leq \operatorname{pdim}_R(k)$

$$\begin{array}{ccc} ① & \Leftrightarrow & ② \\ \Uparrow & & \\ ③ & & \end{array}$$

③ $\Rightarrow$ ① the Koszul complex on a minimal generating set for m is a minimal free resolution for k

Sidenote: Reminders

Prime Avoidance

$P_1, \dots, P_n$ prime ideals $\quad I$ ideal

$I \subseteq P_1 \cup \dots \cup P_n \Rightarrow I \subseteq P_i$ for some i

Equivalently, $I \not\subseteq P_i$ for all $i \Rightarrow I \not\subseteq \bigcup_{i=1}^{n} P_i$

Fancy **Prime Avoidance**

$P_1, \dots, P_n$ prime ideals $\quad x \in R$, I ideal

$(x) + I \not\subseteq P_i$ for all $i \Rightarrow \exists y \in I \quad x + y \notin \bigcup_{i=1}^{n} P_i$

$$① \Leftrightarrow ②$$
$$\Uparrow$$
$$③ \Leftarrow ④$$

④ $\Rightarrow$ ③ $\quad \mathfrak{m} = (x_1, \dots, x_d)$

<u>Claim</u> $(0) \subsetneq (x_1) \subsetneq (x_1, x_2) \subsetneq \cdots \subsetneq (x_1, \dots, x_d)$ are distinct primes

($\Rightarrow x_1, \dots, x_d$ is a regular sequence)

<u>Induction</u> $d = 0$: $\quad \mathfrak{m} = (\{\}) = (0)$ $\quad$ nothing to show

<u>$d > 0$</u> $\quad \mathfrak{m}$ not a minimal prime

Prime Avoidance $\Rightarrow \quad \mathfrak{m} \not\subseteq \bigcup_{P \in \mathrm{Min}(R)} P$

can find $\quad y_1 = x_1 + r_2 x_2 + \cdots + r_d x_d \notin \bigcup_{P \in \mathrm{Min}(R)} P$

$$\mathfrak{m} = (y_1, x_2, \dots, x_d)$$

$\Rightarrow$ can assume $x_1 = y_1$ not contained in any minimal prime

Krull's Height Theorem $\Rightarrow \mathrm{ht}(x_1) \leq 1 \Rightarrow \dim(R/(x_1)) \geq d-1$

Krull's Height theorem $\Rightarrow$ height$\left(\mathfrak{m}/(x_1)\right) \leq d-1$

$$\Rightarrow \dim(R/(x_1)) = d-1$$

Induction Hypothesis $\Rightarrow \quad (x_1)/(x_1), \; (x_1, x_2)/(x_1), \dots, \dfrac{(x_1, \dots, x_d)}{(x_1)}$ distinct primes

$\Rightarrow \quad (x_1) \subsetneq (x_1, x_2) \subsetneq \cdots \subsetneq (x_1, \dots, x_d)$ $\quad$ distinct primes

<u>Claim</u> R is a domain $\quad (\Rightarrow (0)$ also prime$)$

$$\left.\begin{array}{l} x_1 \notin \text{minimal prime} \\ (x_1) \text{ prime} \end{array}\right\} \Longrightarrow \exists \text{ prime } \mathfrak{P} \subsetneq (x_1)$$

$$y \in \mathfrak{P} \subseteq (x_1) \Rightarrow y = r x_1 \xRightarrow[y \in \mathfrak{P}]{x_1 \notin \mathfrak{P}} r \in \mathfrak{P}$$

$$\therefore \quad \mathfrak{P} = \mathfrak{P}(x_1)$$

$$\xRightarrow[NAK]{} \mathfrak{P} = 0$$

$$\begin{array}{ccc} ① & \Leftrightarrow & ② \\ \Uparrow & \searrow & \\ ③ & \Leftarrow & ④ \end{array}$$

① $\Rightarrow$ ④

<u>Claim</u> $\operatorname{pdim}_R(k) < \infty \Longrightarrow \operatorname{pdim}_R(k) \leq d.$

Assume the claim holds. then

$$\underbrace{\beta_i(k)}_{\substack{=0 \\ \text{for } i>d}} = \dim_k \operatorname{Tor}_i^R(k, k) \geq \underbrace{\binom{\mu(m)}{i}}_{\substack{=0 \\ \text{for } i>d}} \quad \text{for all } i$$

$$\Longrightarrow$$

$$\Rightarrow \mu(m) \leq d$$

But $\dim(R) = \text{height}(m) \leq \mu(m) = d$

$$\therefore \quad \mu(m) = d$$

<u>Proof of claim</u> $\operatorname{pdim}_R(k) < \infty \Rightarrow \operatorname{pdim}_R(k) \leq d$.

Assume $\operatorname{pdim}_R(k) > d$

$y_1, \dots, y_t$ maximal regular sequence on $R \Rightarrow t \leq d$

$\Rightarrow$ Every element in $\mathfrak{m}$ is a zerodivisor on $R/(y_1, \dots, y_t)$

$\Rightarrow \mathfrak{m} \in \operatorname{Ass}\left(R/(y_1, \dots, y_t)\right)$

$$R/\mathfrak{m} = k \hookrightarrow R/(y_1, \dots, y_t)$$

ses $\quad 0 \to k \to R/(y_1, \dots, y_t) \to M \to 0$

$$\Big\downarrow \; - \otimes_R k$$

$$\cdots \to \operatorname{Tor}_{i+1}^R(M, k) \to \operatorname{Tor}_i^R(k, k) \to \operatorname{Tor}_i^R\left(R/(y_1, \dots, y_t), k\right) \to \cdots$$

$$\operatorname{pdim}_R\left(R/(y_1, \dots, y_t)\right) = t \leq d < \operatorname{pdim}_R(k)$$

$$\cdots \to \underbrace{\operatorname{Tor}_{i+1}^R(M, k)}_{\substack{=0 \\ \text{for } i = \operatorname{pdim}_R(k)}} \to \operatorname{Tor}_i^R(k, k) \to \underbrace{\operatorname{Tor}_i^R\left(R/(y_1, \dots, y_t), k\right)}_{\substack{=0 \\ \text{for } i = \operatorname{pdim}_R(k)}} \to \cdots$$

But $\operatorname{Tor}_{\operatorname{pdim}_R(k)}^R(k, k) \neq 0$ (!) $\quad ↯$

$\boxtimes$

Corollaries

- Every regular local ring is a domain
- R RLR $\implies \operatorname{pdim}_R(k) = \dim(R)$

Corollary R regular local ring $\implies R_P$ regular

Exercise Show that every PID is a regular ring

R Noetherian ring

Depth I ideal $\quad M$ R-module

theorem All maximal regular sequences on M inside I have the same length

the I-depth of M is

$\operatorname{depth}_I M :=$ length of a maximal regular sequence on M inside I

$\operatorname{depth} M := \operatorname{depth}_m M \quad$ when (R, m) is local

Remark $\operatorname{depth} R \leq \dim R$

theorem R Noetherian, M fg R-module, I ideal, $\underline{x}$ with $(\underline{x})M \neq M$

$$\operatorname{depth}_I(M) := \min\{i \mid \operatorname{Ext}^i_R(R/I, M) \neq 0\}$$

$$\operatorname{depth}_{(x)}(M) := \max\{r \mid H^i(\underline{x}; M) = 0 \text{ for all } i > n - r\}$$

$$\begin{array}{ll} K(\underline{x}; M): & 0 \to R \to R^n \to \cdots \to R^{\binom{n+1}{i}} \to R^{\binom{x}{i}} \to \cdots \\ H(\underline{x}; M): & \underbrace{\qquad\qquad\qquad\qquad\qquad\qquad}_{0} \qquad\qquad \neq 0 \end{array}$$

depth$(R) \le \dim(R)$

More generally, depth$(M) \le \dim(R)$

Cohen-Macaulay ring (R, m) Noetherian local ring

R is Cohen-Macaulay if depth$(R) = \dim(R)$

$$\text{depth } R \underset{\substack{\downarrow \\ = \\ \text{Cohen-Macaulay}}}{\le} \dim(R) \underset{\substack{\downarrow \\ = \\ \text{regular}}}{\le} \text{embdim}(R)$$

M is Cohen-Macaulay if depth(M) = depth(M)

Examples

- Every regular ring is Cohen-Macaulay
- Every 1-dimensional domain is Cohen-Macaulay
- $k[x]/(x^2)$ is Cohen-Macaulay but not regular
- $k[x,y,z]/(xy,yz)$ is not Cohen-Macaulay
- Rings with nice singularities are Cohen-Macaulay

eg

rings of invariants of a finite group G acting on $k[x_1, \ldots, x_d]$, char $k \nmid |G|$

Facts about Cohen-Macaulay rings and modules:

- $\mathfrak{P} \in \mathrm{Ass}(M) \Rightarrow \mathrm{depth}(M) \leq \dim(R/\mathfrak{P})$

- M Cohen-Macaulay $\Rightarrow \mathrm{depth}(M) = \dim(R/\mathfrak{P})$ for all $\mathfrak{P} \in \mathrm{Ass}(M)$
  $\Rightarrow M$ has no embedded primes

- Cohen-Macaulayness localizes

- R Cohen-Macaulay
  $\mathfrak{P}$ prime

$\dim(R) - \mathrm{height}(\mathfrak{P}) = \dim(R/\mathfrak{P})$