Last time **Yoneda Lemma**

$\mathcal{C}$ locally small category

A object in $\mathcal{C}$

$F : \mathcal{C} \to \text{Set}$ covariant functor

then there is a bijection $\text{Nat}(\text{Hom}_{\mathcal{C}}(A, -), F) \xrightarrow{\cong} F(A)$

Consequence Set $F = \text{Hom}_{\mathcal{C}}(B, -)$. then

$$\text{Nat}(\text{Hom}_{\mathcal{C}}(A, -), \text{Hom}_{\mathcal{C}}(B, -)) \xrightarrow{\cong} \text{Hom}_{\mathcal{C}}(B, A)$$

Theorem (Yoneda Embedding) $\mathcal{C}$ locally small category

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \text{Set}^{\mathcal{C}^{op}} \\ A & \longmapsto & \text{Hom}_{\mathcal{C}}(-, A) \end{array} \qquad \begin{array}{ccc} \mathcal{C}^{op} & \longrightarrow & \text{Set}^{\mathcal{C}} \\ A & \longmapsto & \text{Hom}_{\mathcal{C}}(-, A) \end{array}$$

$$\begin{array}{ccc} A & & \text{Hom}_{\mathcal{C}}(-, A) \\ f \downarrow & \longmapsto & \downarrow f_* \\ B & & \text{Hom}_{\mathcal{C}}(-, B) \end{array} \qquad \begin{array}{ccc} A & & \text{Hom}_{\mathcal{C}}(A, -) \\ f \downarrow & \longmapsto & \downarrow f^* \\ B & & \text{Hom}_{\mathcal{C}}(B, -) \end{array}$$

are embeddings.

Slogan Every locally small category embeds into a functor category to set

- A covariant functor $F : \mathcal{C} \longrightarrow \text{Set}$ is <u>representable</u> if it is naturally isomorphic to $\text{Hom}_{\mathcal{C}}(A, -)$ for some A

- A contravariant functor $F : \mathcal{C} \longrightarrow \text{Set}$ is <u>representable</u> if it is naturally isomorphic to $\text{Hom}_{\mathcal{C}}(-, B)$ for some B

<u>Ex</u> $\text{Id} : \text{Set} \longrightarrow \text{Set}$ is represented by 1 (singleton)

Why? X set

$$\begin{array}{ccccc} X & \cong & \text{functions } 1 \longrightarrow X & \equiv & \text{Hom}_{\text{Set}}(1, X) \\ x & \longmapsto & 1 \longmapsto x & & \end{array}$$

the following commutes:

$$\begin{tikzcd} \text{Hom}_{\text{Set}}(1, X) \arrow[r, "\cong"] \arrow[d, "f_*"'] & X \arrow[d, "f"] \\ \text{Hom}_{\text{Set}}(1, Y) \arrow[r, "\cong"'] & Y \end{tikzcd}$$

so our natural isomorphism is

$$X \longmapsto \left(\text{Hom}_{\text{Set}}(1, X) \xrightarrow{\cong} X \right)$$

Complex map / chain map

A map of complexes $f: F \longrightarrow G$ between complexes F, G is a sequence of R-module homomorphisms $f_n: F_n \longrightarrow G_n$ such that the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & F_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "f_{n+1}"'] & F_n \arrow[r, "d_n"] \arrow[d, "f_n"'] & F_{n-1} \arrow[r] & \cdots \\
\cdots \arrow[r] & G_{n+1} \arrow[r, "d_{n+1}"'] & G_n \arrow[r, "d_n"'] & G_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Commutes.

so

$$f_n d_{n+1} = d_{n+1} f_{n+1}$$

Ex: 0 map $\quad 0: F_n \longrightarrow G_n$ for all n

identity: $\quad F_n \xrightarrow{=} F_n$ for all n

Def the category of chain complexes of R-modules

$Ch(R)$ or $Ch(R\text{-mod})$ has:

- objects all complexes of R-modules
- arrows all maps of complex

eg $\quad Ch(Ab) \equiv Ch(\mathbb{Z})$

<u>Lemma</u> $h: F \to G$ map of complexes

then h induces R-module homomorphisms

$$B_n(h): B_n(F) \longrightarrow B_n(G)$$
$$Z_n(h): Z_n(F) \longrightarrow Z_n(G)$$

(<u>recall</u>: $H_n(F) = \dfrac{Z_n(F)}{B_n(F)}$)

$$\Longrightarrow H_n(h): H_n(F) \longrightarrow H_n(G)$$

<u>Proof</u>

$$\begin{tikzcd} F_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "h_{n+1}"'] & F_n \arrow[r, "d_n"] \arrow[d, "h_n"] & F_{n-1} \arrow[d, "h_{n-1}"] \\ G_{n+1} \arrow[r, "d_{n+1}"'] & G_n \arrow[r, "d_n"'] & G_{n-1} \end{tikzcd}$$

- If $a \in B_n(F) = \text{im}\, d_{n+1}^F$, say $a = d_{n+1}(b)$, $b \in F_{n+1}$,

$$h_n(a) = h_n d_{n+1}(b) = d_{n+1} h_{n+1}(b) \in \text{im}\, d_{n+1}^G = B_n(G)$$

- If $a \in Z_n(F) = \ker d_n^F$, then

$$d_n h_n(a) = h_{n-1} \underbrace{d_n(a)}_{=0} = h_{n-1}(0) = 0$$

$$\Rightarrow h_n(a) \in \ker d_n^G = Z_n(G)$$

$$\therefore \quad H_n(F) = Z_n(F)/B_n(F) \longrightarrow Z_n(G)/B_n(G)$$

$\therefore \quad H_n : Ch(R) \longrightarrow R\text{-mod}$ is a functor!

$$F \longmapsto H_n(F)$$

$$\begin{tikzcd} F \arrow[d, "f"'] \\ G \end{tikzcd} \longmapsto \begin{tikzcd} H_n(F) \arrow[d, "H_n(f)"] \\ H_n(G) \end{tikzcd}$$

Def A map of complexes is a quasi-iso if it induces an iso in homology, so $H_n(f)$ is an iso for all n.

Ex:

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "\cdot 2"] \arrow[d, "0"'] & \mathbb{Z} \arrow[r] \arrow[d, "\pi"] & 0 \arrow[d, "0"] \\ 0 \arrow[r] & \mathbb{Z}/2 \arrow[r] & 0 \end{tikzcd}$$

is a quasi-iso but not an iso.

Def $f, g : F \to G$ maps of complexes

A homotopy h between f and g is a sequence of maps

$$h_n : F_n \longrightarrow G_{n+1}$$

$$\begin{tikzcd} \cdots \arrow[r] & F_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, shift right, "f_{n+1}"'] \arrow[d, shift left, "g_{n+1}"] & F_n \arrow[r] \arrow[d, shift right, "f_n"'] \arrow[d, shift left, "g_n"] \arrow[dl, "h_n"'] & F_{n-1} \arrow[r] \arrow[d, shift right, "f_{n-1}"'] \arrow[d, shift left, "g_{n-1}"] \arrow[dl, "h_{n-1}"'] & \cdots \\ \cdots \arrow[r] & G_{n+1} \arrow[r, "d_{n+1}"'] & G_n \arrow[r] & G_{n-1} \arrow[r] & \cdots \end{tikzcd}$$

such that $d_{n+1} h_n + h_{n-1} d_n = f_n - g_n$ for all n.

We say f and g are <u>homotopic</u>

<u>Exercise</u>: Homotopy is an equivalence relation.

<u>Lemma</u> Homotopic maps induce the same map on homology

<u>Proof</u> h homotopy between f and $g : F \to G$

$$a \in Z_n(F) \Rightarrow f_n(a) - g_n(a) = d_{n+1} h_n(a) + h_{n-1} \underbrace{d_n(a)}_{=0} \in B_n(F)$$

$\therefore$ $f_n - g_n$ is 0 in homology!

$$H(f-g) = 0 \Rightarrow H(f) = H(g) \qquad \blacksquare$$

<u>Null homotopic</u> $\equiv$ homotopic to $0 \Rightarrow$ induce 0-map on homology

$f : F \to G$ is a <u>Homotopy Equivalence</u> if there exists some $g : G \to F$ such that fg and gf are homotopic to the identity
(so a homotopy equivalence has an inverse up to homotopy)

homotopy equivalence $\Rightarrow$ quasi-isomorphism

Why? $H_n(f) H_n(g) = H_n(fg) = H_n(id) = id$

$\Rightarrow H_n(f), H_n(g)$ are isos of R-modules

<u>Warning</u> quasi-iso $\not\Rightarrow$ homotopy equivalence.