- F is a subcomplex of G if
  - F_n is a submodule of G_n
  - $$\begin{tikzcd} \cdots \arrow[r] & F_{n+1} \arrow[r] \arrow[d, hook] & F_n \arrow[r] \arrow[d, hook] & \cdots \\ \cdots \arrow[r] & G_{n+1} \arrow[r] & G_n \arrow[r] & \cdots \end{tikzcd}$$ is a map of complexes

the quotient of G by F is the complex

$$\cdots \longrightarrow G_{n+1}/F_{n+1} \xrightarrow{d_{n+1}} G_n/F_n \xrightarrow{d_n} G_{n-1}/F_{n-1} \longrightarrow \cdots$$

where d_n is the map induced by the differential on G.

$f : F \longrightarrow G$ map of complexes

the kernel of f, $\ker f$, is the subcomplex of F

$$\cdots \longrightarrow \ker f_{n+1} \xrightarrow{d_{n+1}} \ker f_n \xrightarrow{d_n} \ker f_{n-1} \longrightarrow \cdots$$

the image of f, $\operatorname{im} f$, is the subcomplex of G

$$\cdots \longrightarrow \operatorname{im} f_{n+1} \xrightarrow{d_{n+1}} \operatorname{im} f_n \xrightarrow{d_n} \operatorname{im} f_{n-1} \longrightarrow \cdots$$

the cokernel of f is the quotient complex

$$\operatorname{coker} f := G/\operatorname{im} f$$

A <u>complex</u> in $Ch(R)$ is a sequence of complex maps

$$\cdots \longrightarrow C^n \xrightarrow{d_n} C^{n-1} \xrightarrow{d_{n-1}} \cdots$$

where $d_{n-1} d_n = 0$ for all n.

A short exact sequence (ses) in $Ch(R)$ is an exact complex

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \to 0 \qquad \text{in } Ch(R)$$

so really, a commutative diagram

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & A_{i+1} \arrow[r, "f_{i+1}"] \arrow[d] & B_{i+1} \arrow[r, "g_{i+1}"] \arrow[d] & C_{i+1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_i \arrow[r, "f_i"] \arrow[d] & B_i \arrow[r, "g_i"] \arrow[d] & C_i \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_{i-1} \arrow[r, "f_{i-1}"] \arrow[d] & B_{i-1} \arrow[r, "g_{i-1}"] \arrow[d] & C_{i-1} \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

where the rows are exact and the columns are complexes.

Snake Lemma

A commutative diagram of R-modules with exact rows.

$$\begin{tikzcd}
 & A' \arrow[r, "i'"] \arrow[d, "f"'] & B' \arrow[r, "p'"] \arrow[d, "g"'] & C' \arrow[r] \arrow[d, "h"'] & 0 \\
0 \arrow[r] & A \arrow[r, "i"'] & B \arrow[r, "p"'] & C &
\end{tikzcd}$$

induces an exact sequence

$$\ker f \to \ker g \to \ker h \xrightarrow{\partial} \operatorname{coker} f \to \operatorname{coker} g \to \operatorname{coker} h$$

$$\begin{tikzcd}
 & \ker f \arrow[r, dashed] \arrow[d] & \ker g \arrow[r, dashed] \arrow[d] & \ker h \arrow[d] \arrow[dddll, bend left, "\partial"] \\
 & A' \arrow[r, "i'"] \arrow[d, "f"'] & B' \arrow[r, "p'"] \arrow[d, "g"'] & C' \arrow[d, "h"] \\
0 \arrow[r] & A \arrow[r, "i"'] \arrow[d] & B \arrow[r, "p"'] \arrow[d] & C \arrow[d] \\
 & \operatorname{coker} f \arrow[r, dashed] & \operatorname{coker} g \arrow[r, dashed] & \operatorname{coker} h
\end{tikzcd}$$

$$\partial(c') := a + \operatorname{im} f \in \operatorname{coker} f$$

let $b' \in B'$ be such that $p'(b') = c'$ (p is surjective)

let $a \in A$ be such that $i(a) = g(b')$

∂ is called the connecting homomorphism.

① $\ker f \longrightarrow \ker g \longrightarrow \ker h$

are restrictions of $A' \xrightarrow{i'} B' \xrightarrow{p'} C'$

and exactness in the middle is preserved

② $A \xrightarrow{i} B \xrightarrow{p} C$ restrict to

$\operatorname{im} f \longrightarrow \operatorname{im} g \longrightarrow \operatorname{im} h$ so they descend to

$\operatorname{coker} f \longrightarrow \operatorname{coker} g \longrightarrow \operatorname{coker} h$

and exactness in the middle is preserved.

Need: ③ ∂ is well-defined

④ $p'(\ker g) = \ker \partial$

⑤ $\operatorname{im} \partial = \ker(\operatorname{coker} f \xrightarrow{i} \operatorname{coker} g)$

$$\begin{tikzcd}
 & & c' \in \ker h \arrow[d, mapsto] & \\
A' \arrow[d, "f"'] & b' \in B' \arrow[r, mapsto, two heads, "p'"] \arrow[d, mapsto, "g"] & c' \in C' \arrow[d, mapsto] & \\
a \in A \arrow[r] \arrow[d, mapsto] & g(b') \in B \arrow[r, mapsto, "p"'] & 0 \arrow[r, Rightarrow] & g(b') \in \ker p = \operatorname{im} i \\
a + \operatorname{im} f \in \operatorname{coker} f & & &
\end{tikzcd}$$

③ fix $c' \in \ker h$, $b_1', b_2' \in B'$ such that $p(b_1') = p(b_2') = c'$.
Want to check: ∂ does not depend on b_1', b_2'.

<u>Note</u>: $p'(b_1'-b_2') = 0$, so it's enough to check $\partial(0)=0$ independently of the choice of $b'\in B'$ with $p'(b')=0$

$$b'\in \ker p' = \operatorname{im} i' \implies i(a') = b' \text{ for some } a'\in A'$$

then $a := f(a')$ satisfies

$$i(a) = i(f(a')) = gi'(a') = g(b')$$

so $\partial(0) = a + \operatorname{im} f = f(a') + \operatorname{im} f = 0$ in $\operatorname{coker} f$

④ $p'(\ker g) = \ker \partial$

• If $b'\in \ker g$, then
$$\begin{tikzcd} & b' \arrow[d, mapsto, "g"] \\ 0 \arrow[r, mapsto, "i"] & 0 \end{tikzcd} \implies \partial(p'(b')) = 0$$

• If $c'\in C'$ has $\partial(c')=0$, let $b'\in B'$ with $p'(b')=c'$

$$\begin{tikzcd} a'\in A' \arrow[d, dashed, "f"'] & b' \arrow[r, mapsto, "p'"] \arrow[d, mapsto, "g"] & c' \\ a\in \operatorname{im} f \arrow[r, mapsto, "i"] & g(b') & \end{tikzcd}$$

Must have $a\in \operatorname{im} f \implies$ let $a'\in A'$ be such that $f(a') = a$
then
$$gi'(a') = if(a') = i(a) = g(b')$$
$$\implies b' - i'(a') \in \ker g, \text{ and}$$
$$p'(b'-i'(a')) = p'(b') - \underbrace{p'i'(a')}_{=0} = p'(b') = c'$$
$$\therefore c'\in p'(\ker g)$$

⑤ $\operatorname{im} \partial = \ker(\operatorname{coker} f \xrightarrow{i} \operatorname{coker} g)$

- Given $a \in A$, if $a + \operatorname{im} f \in \ker(\operatorname{coker} f \xrightarrow{i} \operatorname{coker} g)$

$i(a + \operatorname{im} f) = 0 \Rightarrow i(a) \in \operatorname{im} g \Rightarrow i(a) = g(b')$ for some $b' \in B$

so $\partial(p(b')) \underset{\text{by definition}}{=} a + \operatorname{im} f$

$\Rightarrow a + \operatorname{im} f \in \operatorname{im} \partial$

- Given $a + \operatorname{im} f \in \operatorname{im} \partial$,

$$\begin{tikzcd} & b' \arrow[r, mapsto] \arrow[d, mapsto] & c' \\ a \arrow[r, mapsto] & g(b') & \end{tikzcd}$$

$i(a + \operatorname{im} f) = i(a) + \operatorname{im} g = g(b') + \operatorname{im} g = 0$ in $\operatorname{coker} g$

$\square$

<u>theorem</u> (Long Exact sequence in homology)

Given a short exact sequence in $Ch(R)$

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

there is a long exact sequence in R-mod

$$\cdots \to H_{n+1}(C) \xrightarrow{\partial} H_n(A) \to H_n(B) \to H_n(C) \xrightarrow{\partial} \cdots$$

Proof For each n, we have short exact sequences

$$0 \longrightarrow A_n \xrightarrow{f_n} B_n \xrightarrow{g_n} C_n \longrightarrow 0$$

- f, g take cycles to cycles, so we get

$$0 \longrightarrow Z_n(A) \xrightarrow{f_n} Z_n(B) \xrightarrow{g_n} Z_n(C) \quad \underline{\text{exact}}.$$

- f, g take boundaries to boundaries, so get exact

$$A_n/\operatorname{im} d_{n+1} \xrightarrow{f_n} B_n/\operatorname{im} d_{n+1} \xrightarrow{g_n} C_n/\operatorname{im} d_{n+1} \longrightarrow 0$$

Let $F = A, B$, or C.

$$\operatorname{im}\left(F_n \xrightarrow{d_n} F_{n-1}\right) \subseteq Z_{n-1}(F)$$

so induces

$$F_n \xrightarrow{d_n} Z_{n-1}(F)$$

that sends $\operatorname{im} d_{n+1}$ to 0, so get an induced map

$$F_n/\operatorname{im} d_{n+1} \xrightarrow{d_n} Z_{n-1}(F)$$

so:

$$\begin{tikzcd}
& A_n/\operatorname{im} d_{n+1} \arrow[r] \arrow[d, "d_n"'] & B_n/\operatorname{im} d_{n+1} \arrow[r] \arrow[d, "d_n"'] & C_n/\operatorname{im} d_{n+1} \arrow[r] \arrow[d, "d_n"'] & 0 \\
0 \arrow[r] & Z_{n-1}(A) \arrow[r] & Z_{n-1}(B) \arrow[r] & Z_{n-2}(C) &
\end{tikzcd}$$

Apply the Snake lemma:

$$\ker\left(F_n/\operatorname{im} d_{n+1} \xrightarrow{d_n} Z_{n-1}(F) \right) = \frac{\ker d_n}{\operatorname{im} d_{n+1}} = H_n(F)$$

$$\operatorname{coker}\left(F_n/\operatorname{im} d_{n+1} \xrightarrow{d_n} Z_{n-1}(F) \right) = \frac{Z_{n-1}(F)}{\operatorname{im} d_n} = H_{n-1}(F)$$

so we get exact sequences

$$H_n(A) \to H_n(B) \to H_n(C) \xrightarrow{\partial} H_{n-1}(A) \to H_{n-1}(B) \to H_{n-1}(C)$$

glue these together. ∎

Connecting homomorphism:

$$c \in \ker d_{n+1}^{C} \subseteq C_{n+1} \Rightarrow \exists\, b \in B_{n+1} \quad \underbrace{g_{n+1}}_{\text{surjective}}(b) = c$$

$$\begin{array}{ccc} & b & \overset{g_{n+1}}{\longmapsto} \; c \\ & \downarrow{\scriptstyle d_{n+1}} & \\ a \overset{f_n}{\longmapsto} & d_{n+1}(b) & \end{array}$$

$$\partial(c) = a + \operatorname{im} d_{n+1}^{A}$$