Last time

Long Exact Sequence in homology

A short exact sequence in $Ch(R)$

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

induces a long exact sequence

$$\cdots \to H_{n+1}(C) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{p_*} H_n(A) \xrightarrow{\partial} H_{n-1}(C) \to \cdots$$

Given $c \in Z_{n+1}(C) = \ker(d^C_{n+1} : C_{n+1} \to C_n)$

$$\begin{tikzcd} & b \in B_{n+1} \arrow[r, two heads] \arrow[d, "d^B_{n+1}"'] & c \in C_{n+1} \\ a \in A_n \arrow[r, mapsto] & d^B_{n+1}(b) & \end{tikzcd}$$

$$\partial(c) = a + \operatorname{im} d^A_{n+1} \in H_{n+1}(A)$$

<u>theorem</u> Any commutative diagram in $Ch(R)$

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "f"] & B \arrow[r, "p"] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & A' \arrow[r, "i'"'] & B' \arrow[r, "p'"'] & C' \arrow[r] & 0 \end{tikzcd}$$

with exact rows induces a commutative diagram with exact rows

$$\begin{tikzcd} \cdots \arrow[r] & H_{n+1}(C) \arrow[r, "\partial"] \arrow[d, "h_*"] & H_n(A) \arrow[r, "i_*"] \arrow[d, "f_*"] & H_n(B) \arrow[r, "p_*"] \arrow[d, "g_*"] & H_n(C) \arrow[r, "\partial"] \arrow[d, "h_*"] & H_{n-1}(A) \arrow[r] \arrow[d, "f_*"] & \cdots \\ \cdots \arrow[r] & H_{n+1}(C') \arrow[r, "\partial'"'] & H_n(A') \arrow[r, "i'_*"'] & H_n(B') \arrow[r, "p'_*"'] & H_n(C') \arrow[r, "\partial'"'] & H_{n-1}(A') \arrow[r] & \cdots \end{tikzcd}$$

Proof LES in homology $\Rightarrow$ exact rows

H_n functoual $\Rightarrow$

$$\begin{tikzcd}
H_n(A) \arrow[r, "i_*"] \arrow[d, "f_*"'] & H_n(B) \arrow[r, "p_*"] \arrow[d, "g_*"'] & H_n(C) \arrow[d, "h_*"'] \\
H_n(A') \arrow[r, "i'_*"'] & H_n(B') \arrow[r, "p'_*"'] & H_n(C')
\end{tikzcd}$$

commute

Need to show:

$$\begin{tikzcd}
H_n(C) \arrow[r, "\partial"] \arrow[d, "h_*"'] & H_{n-1}(A) \arrow[d, "f_*"] \\
H_n(C') \arrow[r, "\partial'"'] & H_{n-1}(A')
\end{tikzcd}$$

commutes

Let $c \in \ker(d_n : C_n \longrightarrow C_{n-1})$. Need:

(L) $f_{n-1}\,\partial(c) = \partial' h_n(c)$ (R)

$$\begin{tikzcd}
0 \arrow[r] & A_n \arrow[r, "i_n"] \arrow[d, "d_n"'] \arrow[dd, dashed, bend left] & B_n \arrow[r, "p_n"] \arrow[d, "d_n"'] \arrow[dd, dashed, bend left] & C_n \arrow[r] \arrow[d, "d_n"'] \arrow[dd, dashed, bend left] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r, "i_{n-1}"] \arrow[dd, bend right] & B_{n-1} \arrow[r, "p_{n-1}"] \arrow[dd, bend right] & C_{n-1} \arrow[r] \arrow[dd, bend right] & 0 \\
0 \arrow[r] & A'_n \arrow[r, "i'_n"] \arrow[d, "d_n"'] & B'_n \arrow[r, "p'_n"] \arrow[d, "d_n"'] & C'_n \arrow[r] \arrow[d, "d_n"'] & 0 \\
0 \arrow[r] & A'_{n-1} \arrow[r, "i_n"'] & B_{n-1} \arrow[r, "p_{n-1}"'] & C_{n-1} \arrow[r] & 0
\end{tikzcd}$$

(L):

$\partial(c) = a + \operatorname{im} d_n \in H_{n-1}(A)$, where $i_{n-1}(a) = d_n(b)$, $p_n(b) = c$

$\Rightarrow f_{n-1}\,\partial(c) = f_{n-1}(a + \operatorname{im} d_n^A) = f_{n-1}(a) + \operatorname{im} d_n^{A'} \in H_{n-1}(A')$

(R): to compute $\partial' h_n(c)$, we:

<u>Step 1</u> Find

$$b' \in B'_n \xmapsto{p'_n} h_n(c) \in C'_n$$

By commutativity of

$$\begin{tikzcd} B_n \arrow[r, "p_n"] \arrow[d, "g_n"'] & C_n \arrow[d, "h_n"] \\ B'_n \arrow[r, "p'_n"'] & C'_n \end{tikzcd}$$

can choose $b' := g_n(b)$ where $p_n(b) = c$ (from (2))

because $p'_n(b') = p'_n g_n(b) = h_n p_n(b) = h_n(c)$

<u>Step 2</u> take $a' \in A'_{n-1}$ such that

$$\begin{tikzcd} & b' \in B'_n \arrow[r, mapsto, "p'_n"] \arrow[d, "d_n"] & h_n(c) \\ a' \in A'_{n-1} \arrow[r, mapsto] & d_n(b') & \end{tikzcd}$$

set $\partial' h_n(c) := a' + \operatorname{im} d_n \in H_{n-1}(A')$

By commutativity of

$$\begin{tikzcd} B_n \arrow[r, "d_n"] \arrow[d, "g_n"'] & B_{n-1} \arrow[d, "g_{n-1}"] \\ B'_n \arrow[r, "d_n"'] & B'_{n-1} \end{tikzcd}$$

$$d_n(b') = d_n g_n(b) = g_{n-1} d_n(b)$$

Our choice of a in ① satisfies

$$d_n(b') = g_{n-1} d_n(b) = g_{n-1}\, i_{n-1}(a)$$

and by commutativity of

$$\begin{tikzcd} A_{n-1} \arrow[r, "i_{n-1}"] \arrow[d, "f_{n-1}"'] & B_{n-1} \arrow[d, "g_{n-1}"] \\ A'_{n-1} \arrow[r, "i'_{n-1}"'] & B'_{n-1} \end{tikzcd}$$

we have

$$i'_{n-1} f_{n-1}(a) = g_{n-1}\, i_{n-1}(a) \overset{\text{above}}{=} d_n(b')$$

so $f_{n-1}(a)$ satisfies

$$i'_{n-1} f_{n-1}(a) = d_n(b')$$

so can choose $a' := f_{n-1}(a)$, and

$$\partial' h_n(c) = a' + \operatorname{im} d_n = f_{n-1}(a) + \operatorname{im} d_n \in H_{n-1}(A')$$

so: $\partial' h_n(c) = f_{n-1}\, \partial(c)$ as we wanted !

$\blacksquare$

Comments :

① the LES in homology can often be used to find the homology of a particular complex by fitting it into a SES of complexes with some well-known or easier to understand complexes.

② Later we will see how LES naturally pop in everywhere. In particular, we will study derived functors such as Tor and Ext, which are constructed via the homology of a particular complex, and which must then induce a LES from any SES.

③ Any LES breaks into SES:

$$\cdots \longrightarrow C_{n+1} \xrightarrow{f_{n+1}} C_n \xrightarrow{f_n} C_{n-1} \longrightarrow \cdots \qquad \text{LES}$$

breaks into

$$\begin{tikzcd}
0 \arrow[dr] & & & & \\
& \ker f_n \arrow[dr] & & & \\
& C_{n+1} \arrow[r, "f_{n+1}"'] & C_n \arrow[r, "f_n"'] \arrow[dr] & C_{n-1} & \\
& & & \operatorname{coker} f_{n+1} \arrow[dr] & \\
& & & & 0
\end{tikzcd}$$

$$\ker f_n = \operatorname{im} f_{n+1} = \ker(C_n \to \operatorname{coker} f_n)$$

so

$$0 \to \ker f_n \to C_n \to \operatorname{coker} f_{n+1} \to 0$$

is exact.

<u>R-mod</u> category with

- objects R-modules
- arrows R-module homomorphisms

<u>Notation</u> $\mathrm{Hom}_R(M,N) := \mathrm{Hom}_{R\text{-mod}}(M,N)$

this is a locally small category, so $\mathrm{Hom}_R(M,N)$ is a set.
It also has <u>more</u> additional structure

Given R-modules M, N, $\mathrm{Hom}_R(M,N)$ is an R-module via

$$\underset{R}{\underset{\cap}{r}} \cdot \underset{\mathrm{Hom}_R(M,N)}{\underset{\cap}{f}} := \big(m \longmapsto r f(m) \big)$$

with addition $(f+g)(m) = f(m) + g(m)$

0: 0-map

<u>Exercise</u> Show $\mathrm{Hom}_R(M,N)$ is indeed an R-module.

<u>Exercise</u>

① $\mathrm{Hom}_R(R,M) \cong M$

② $\mathrm{Hom}_R(R/I, M) \cong (0 :_M I) = \{ m \in M \mid Im = 0 \}$

$$\left.\begin{array}{l}\mathrm{Hom}_R(M,-)\\ \mathrm{Hom}_R(-,N)\end{array}\right\} : R\text{-mod} \longrightarrow R\text{-mod}$$

are <u>additive</u> functors.

A functor $T: R\text{-mod} \to S\text{-mod}$ is <u>additive</u>

if $T(f+g) = T(f) + T(g)$

for all $f, g \in \mathrm{Hom}_R(M,N)$

<u>Lemma</u> $T: R\text{-mod} \to S\text{-mod}$ additive functor

1. $T(0\text{-map}) = 0\text{-map}$
2. $T(0\text{-module}) = 0\text{-module}$.