Sidenote: Universal properties

We've all seen constructions satisfying a universal property.
these are best explained by examples:

Examples

① Products $\{M_i\}_{i \in I}$ collection of R-modules

the product of the $\{M_i\}_{i\in I}$ is the R-module

$$\prod_{i\in I} M_i = \{ (m_i)_{i\in I} : m_i \in M_i \}$$

With the following universal property:
given an R-module N and R-module homomorphisms $N \xrightarrow{f_i} M_i$
there exists a unique R-module homomorphism f such that

$$\begin{tikzcd} N \arrow[r, dashed, "f"] \arrow[dr, "f_j"'] & \prod_{i\in I} M_i \arrow[d, "\pi_j"] \\ & M_j \end{tikzcd}$$

commutes for all $j \in I$

where $\pi_j : \prod_{i \in I} M_i \longrightarrow M_j$ is the projection onto the jth factor

Slogan Mapping into a product $\Leftrightarrow$ mapping into each factor

② **Direct sum** $\{M_i\}_{i\in I}$ collection of R-modules

the direct sum of the $\{M_i\}_{i\in I}$ is the R-module

$$\bigoplus_{i\in I} M_i = \{(m_i)_{i\in I} : m_i \in M_i,\ m_j = 0 \text{ for all but finitely many } j\}$$

with the following universal property:

given an R-module N and R-module homomorphisms $M_i \xrightarrow{f_i} N$ there exists a unique R-module homomorphism such that

$$\begin{tikzcd} N & \bigoplus_{i\in I} M_i \arrow[l, dashed, "f"'] \\ & M_j \arrow[u, "\iota_j"'] \arrow[ul, "f_j"] \end{tikzcd}$$

commutes for all $j \in I$

where $\iota_j : M_j \longrightarrow \bigoplus_{i\in I} M_i$ is the inclusion of the jth factor

Slogan Mapping out of a direct sum $\Leftrightarrow$ mapping out of each factor

Note When I is finite, $\prod_{i\in I} M_i = \bigoplus_{i\in I} M_i$

and we may write $M_1 \times \cdots \times M_n$ (direct product)

or

$M_1 \oplus \cdots \oplus M_n$ (direct sum)

When I is infinite, the constructions are different.

<u>Free R-module</u> the free R-module on $\{b_i\}_{i\in I}$ is the R-module F with the following universal property: given an R-module M and a function $f: I \longrightarrow M$, there exists a unique R-module homomorphism $\hat{f}: F \longrightarrow M$ such that

$$\begin{tikzcd} i \arrow[r, mapsto] & b_i \\ I \arrow[r] \arrow[dr, "f"'] & F \arrow[d, dashed, "\hat{f}"] \\ & M \end{tikzcd}$$

commutes

ie, $\hat{f}(b_i) = f(i)$.

We say F is the free module on I (or $\{b_i\}_{i\in I}$) and $\{b_i\}$ is called a basis.

<u>Slogan</u> to define a map out of a free module is to define any function (of sets) on the basis elements

<u>Real slogan</u> The basis elements form an actual basis (there are no relations between them, and they generate F)

We saw in the Winter that $F \cong \bigoplus_{i\in I} R = R^{\oplus I}$ or R^I

Product (category theory) $\mathcal{C}$ category

$\{M_i\}_{i\in I}$ a collection of objects in $\mathcal{C}$, indexed by a set I

the product of $\{M_i\}_{i\in I}$ is the object $\prod_{i\in I} M_i$

together with arrows $\prod_{i\in I} M_i \xrightarrow{\pi_j} M_j$

with the following universal property:

given arrows $N \xrightarrow{f_i} M_i$ there exists a unique arrow f

such that
$$\begin{tikzcd} N \arrow[r, dashed, "f"] \arrow[dr, "f_j"'] & \prod_{i\in I} M_i \arrow[d, "\pi_j"] \\ & M_j \end{tikzcd}$$
commutes.

Examples

- the product in R-mod is the product of modules.

- In Top, the product of topological spaces is the set product with the Tychonoff topology

  (with the basis of open sets $\prod_{i\in I} U_i \subseteq \prod_{i\in I} X_i$ where $U_i = X_i$ for all but finitely many i, all U_i open)

- If I is a partially ordered set, when we view I as a category we have products = greatest lower bounds

- In Fields, there is no $\mathbb{Q}\times \mathbb{Z}/p$

Coproduct (category theory) $\mathscr{C}$ category

$\{M_i\}_{i\in I}$ a collection of objects in $\mathscr{C}$, indexed by a set I

the coproduct of $\{M_i\}_{i\in I}$ is the object $\coprod_{i\in I} M_i$

together with arrows $M_j \xrightarrow{\iota_j} \coprod_{i\in I} M_i$

with the following universal property:

given arrows $M_i \xrightarrow{f_i} N$ there exists a unique arrow f

such that
$$\begin{tikzcd} N & \coprod_{i\in I} M_i \arrow[l, dashed, "f"'] \\ & M_j \arrow[u, "\iota_j"'] \arrow[ul, "f_j"] \end{tikzcd}$$
commutes.

Examples

- In R-mod, the coproduct is the direct sum
- In Set, the coproduct is the disjoint union of sets
- In Grp, the coproduct is the free product
- If I is a partially ordered set, and we view it as a category,

  coproducts = least upper bound

Free object (category theory) (Concrete category)

Let $\mathcal{C}$ be a category with a faithful functor $F: \mathcal{C} \longrightarrow \mathrm{Set}$

Let X be a set

the free object on X is an object A in $\mathcal{C}$
together with an injective function of sets $X \xrightarrow{i} F(A)$
that satisfies the following universal property:

for any object B in $\mathcal{C}$ and any function of sets $X \xrightarrow{f} F(B)$
there exists a unique arrow $A \xrightarrow{g} B$ such that

$$
\begin{tikzcd}
A \arrow[d, dashed, "g"'] \\
B
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
X \arrow[r, "i"] \arrow[dr, "f"'] & F(A) \arrow[d, "F(g)"] \\
& F(B)
\end{tikzcd}
$$

Example In R-mod, free modules

When we talk about adjoint functors:

a forgetful functor to Set has a left adjoint that is the free functor.

Universal property (formal definition)
$\mathcal{C}$ locally small category

A universal property for an object c in $\mathcal{C}$ consists of:

- a representable functor $F : \mathcal{C} \longrightarrow \mathrm{Set}$
- a universal element $x \in F(c)$ such that

F is naturally isomorphic to $\begin{matrix} \mathrm{Hom}_{\mathcal{C}}(c, -) \\ \mathrm{Hom}_{\mathcal{C}}(-, c) \end{matrix}$

via the natural iso corresponding to x by the Yoneda bijection.

Example the universal property of the product $X_1 \times X_2$ is encoded in

- the representable functor $\mathrm{Hom}_{\mathcal{C}\times\mathcal{C}}(\Delta(-), (X_1, X_2))$
- represented by $X_1 \times X_2$
- via $(\pi_1, \pi_2) \in \mathrm{Hom}_{\mathcal{C}\times\mathcal{C}}(\Delta(X_1 \times X_2), (X_1, X_2))$

where
$$
\begin{array}{rcl}
\Delta : \mathcal{C} & \longrightarrow & \mathcal{C} \times \mathcal{C} \\
X & & (X, X) \\
f \downarrow & & \downarrow (f, f) \\
Y & & (Y, Y)
\end{array}
$$

Unwrapping this, it says:

$$\mathrm{Hom}_{\mathcal{C}}(-, X_1 \times X_2) \cong \mathrm{Hom}_{\mathcal{C}\times\mathcal{C}}(\Delta(-), (X_1, X_2))$$

by the natural isomorphism φ given by

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{C}}(Y, X_1 \times X_2) & \xrightarrow{\varphi_Y} & \mathrm{Hom}_{\mathcal{C}\times\mathcal{C}}(\Delta(Y), (X_1, X_2)) \\
f & \longmapsto & \Delta(f)^*(\pi_1, \pi_2) \\
 & & \shortparallel \\
\mathrm{id}_Y & & (\pi_1, \pi_2) \circ \Delta(f)
\end{array}$$

$\varphi_Y(f) = (g_1, g_2)$ means

$$\begin{tikzcd}
 & X_1 \times X_2 \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
X_1 & & X_2 \\
 & Y \arrow[ul, "g_1"] \arrow[ur, "g_2"'] \arrow[uu, "f"] &
\end{tikzcd}$$

commutes

φ_Y is a bijection

$\Rightarrow$ for every g_1, g_2 there exists a unique f such that the diagram commutes.

(that's the universal property of the product!)

Back to R-mod:

$\mathrm{Hom}_R(M,-)$ and $\mathrm{Hom}_R(-,N)$ are additive functors R-mod $\longrightarrow$ R-mod

theorem there are natural isomorphisms

$$\mathrm{Hom}_R(M, \prod_i N_i) \cong \prod_i \mathrm{Hom}_R(M, N_i)$$

$$\mathrm{Hom}_R(\bigoplus_i M_i, N) \cong \bigoplus_i \mathrm{Hom}_R(M_i, N_i)$$

T: R-mod $\longrightarrow$ S-mod covariant additive functor

- T is left exact if

$0 \to A \to B \to C$ exact $\Rightarrow$ $0 \to T(A) \to T(B) \to T(C)$ exact

- T is right exact if

$A \to B \to C \to 0$ exact $\Rightarrow$ $T(A) \to T(B) \to T(C) \to 0$ exact

- T is exact if

$0 \to A \to B \to C \to 0$ ses $\Rightarrow$ $0 \to T(A) \to T(B) \to T(C) \to 0$

$T : R\text{-mod} \longrightarrow S\text{-mod}$ Contravariant additive functor

- T is <u>left exact</u> if

$A \to B \to C \to 0$ exact $\Rightarrow$ $0 \to T(C) \to T(B) \to T(A)$ exact

- T is <u>right exact</u> if

$0 \to A \to B \to C$ $\Rightarrow$ $T(C) \to T(B) \to T(A) \to 0$ exact

- T is <u>exact</u> if

$0 \to T(C) \to T(B) \to T(A) \to 0$ exact

<u>Remark</u>:

Equivalently, we can start with a ses in all the conditions

- Left exact = turns kernels into kernels (cokernels)
- Right exact = turns cokernels into cokernels (kernels)
- Exact = left exact + right exact

(contravariant)

theorem (Hom is left exact)

- For every short exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

and every R-module M,

$$0 \longrightarrow \mathrm{Hom}_R(M, A) \xrightarrow{f_*} \mathrm{Hom}_R(M, B) \xrightarrow{g_*} \mathrm{Hom}_R(M, C)$$

is exact

- For every short exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

and every R-module M,

$$0 \longrightarrow \mathrm{Hom}_R(M, C) \xrightarrow{g^*} \mathrm{Hom}_R(M, B) \xrightarrow{f^*} \mathrm{Hom}_R(M, A)$$

is exact

Proof Covariant version:

① f_* is injective

Let $h \in \mathrm{Hom}_R(M, A)$.

$$f_*(h) = 0 \iff f \circ h = 0 \underset{f \text{ mono}}{\overset{f \text{ injective}}{\Longrightarrow}} h = 0$$

② $\operatorname{im} f_* \subseteq \ker g_*$

For all $h \in \operatorname{Hom}_R(M, A)$, $\quad g_* f_*(h) = \underbrace{g \circ f}_{=0} \circ h = 0$

③ $\ker g_* \subseteq \operatorname{im} f_*$

Let $h \in \operatorname{Hom}_R(M, B)$, $\quad g_*(h) = 0 \iff g \circ h = 0$

For all $m \in M$, $g(h(m)) = 0 \Rightarrow h(m) \in \ker g = \operatorname{im} f$

Choose $a \in A$ such that $f(a) = h(m)$, and set

$$k(m) := a.$$

- f injective $\Rightarrow$ for each m $\exists$ unique a $\Rightarrow$ k is well-defined
- Exercise: $k \in \operatorname{Hom}_R(M, A)$
- $f_*(k)$ satisfies $\quad f_*(k)(m) = f(k(m)) = f(a) \overset{\text{def of } a}{=} h(m)$

$\Rightarrow f_*(k) = h.$

$\square$

Warning: Hom is not right exact in general

Example $\quad 0 \to \mathbb{Z} \xrightarrow{i} \mathbb{Q} \xrightarrow{\pi} \mathbb{Q}/\mathbb{Z} \to 0$

$$\Downarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2, -)$$

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Z}) \xrightarrow{i_*} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Q}) \xrightarrow{\pi_*} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Q}/\mathbb{Z})$$

Claim π_* not surjective

- $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Q}) = 0 \quad \left(1 \mapsto a, \quad 2a = 0 \Rightarrow a = 0\right)$
- $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Q}/\mathbb{Z}) \neq 0$

  $1 \mapsto \frac{1}{2} + \mathbb{Z}$ is ok!

so actually, our exact sequence is

$$0 \to 0 \to 0 \to \underbrace{\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Q}/\mathbb{Z})}_{\cong \mathbb{Z}/2}$$

✓

Also, when we apply $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Z})$, we get

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, \mathbb{Z}) \xrightarrow{g^*} \underbrace{\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z})}_{=0} \xrightarrow{f^*} \underbrace{\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z})}_{\mathbb{Z}}$$

$\Rightarrow f^*$ not surjective!