Last time:

$\mathrm{Hom}_R(-, M)$ and $\mathrm{Hom}_R(M, -)$ are additive left exact functors

meaning that for every short exact sequence

$$0 \to A \to B \to C \to 0$$

of R-modules, the sequences

$$0 \to \mathrm{Hom}_R(M, A) \to \mathrm{Hom}_R(M, B) \to \mathrm{Hom}_R(M, C)$$

and

$$0 \to \mathrm{Hom}_R(C, M) \to \mathrm{Hom}_R(B, M) \to \mathrm{Hom}_R(A, M)$$

are exact.

However, $\mathrm{Hom}_R(M, -)$ and $\mathrm{Hom}_R(-, M)$ are not necessarily exact.

Tensor products

M, N R-modules

A function $f: M \times N \to L$ is R-bilinear if:

- $f(m+m', n) = f(m, n) + f(m', n)$ for all $m, m' \in M$, $n \in N$
- $f(m, n+n') = f(m, n) + f(m, n')$ for all $m \in M$, $n, n' \in N$
- $f(rm, n) = f(m, rn) = rf(m, n)$ for all $r \in R$, $m \in M$, $n \in N$

Example the product on R is an R-bilinear function $R \times R \longrightarrow R$

Def M, N R-modules

the tensor product $M \otimes_R N$ is the R-module $M \otimes_R N$ together with an R-bilinear map $\tau : M \times N \longrightarrow M \otimes_R N$ satisfying the following universal property:

for every R-bilinear function $f : M \times N \longrightarrow A$, there exists a unique R-module homomorphism $\hat{f} : M \otimes_R N \longrightarrow A$ such that

$$\begin{tikzcd} M \otimes_R N \arrow[drr, dashed, "\hat{f}"] & & \\ M \times N \arrow[u, "\tau"] \arrow[rr, "f"'] & & A \end{tikzcd}$$

commutes

this universal property is encoded in the representable functor

$$\begin{array}{cccc} \mathrm{Bilin}(M \times N ; -) : & R\text{-mod} & \longrightarrow & \mathrm{Set} \\ & L & \longmapsto & R\text{-bilinear maps } M \times N \longrightarrow L \\ & \begin{matrix} A \\ f \downarrow \\ B \end{matrix} & \longmapsto & \text{postcomposition with } f \end{array}$$

via

$$\tau \in \mathrm{Bilin}(M \times N, M \otimes_R N)$$

<u>thm</u> Given any R-modules M, N, $M \otimes_R N$ exists

<u>Sketch</u>

$F :=$ free module on $M \times N$

$$S := \left(\begin{array}{c} (m, n+n') - (m,n) - (m,n') \\ (m+m', n) - (m,n) - (m',n) \\ (rm, n) - r(m,n) \\ (m, rn) - r(m,n) \end{array} \;\middle|\; \begin{array}{c} m, m' \in M \\ n, n' \in N \\ r \in R \end{array} \right)$$

$M \otimes_R N := F/S$

$$\begin{array}{ccc} M \times N & \xrightarrow{\tau} & F/S \\ (m,n) & \longmapsto & m \otimes n \end{array}$$

$m \otimes n := (m,n) + S$ in F/S

<u>Check</u>: these satisfy our universal property ☒

<u>Def</u> A <u>simple tensor</u> in $M \otimes_R N$ is an element of the form $m \otimes n$.

The simple tensors <u>generate</u> $M \otimes_R N$, but they are <u>not</u> all the elements.

so every element in $M \otimes_R N$ is of the form $\sum_{i=1}^{k} m_i \otimes n_i$

such expressions are <u>not</u> unique.

<u>Warning</u> Sometimes $M \otimes_R N$ are unexpectedly 0

- to define maps out of $M \otimes_R N$, use the universal property.
- to show $M \otimes_R N \neq 0$, give a nonzero R-bilinear map $M \times N \to A$.
- two R-module maps $M \otimes_R N \to A$ are the same if they agree on simple tensors.

<u>Lemma</u> the tensor product is unique up to iso.

<u>Lemma</u> $A \xrightarrow{f} C$, $B \xrightarrow{g} D$ R-module maps
there exists a unique R-module homomorphism

$$\begin{aligned} f \otimes g : A \otimes_R B &\longrightarrow C \otimes_R D \\ a \otimes b &\longmapsto f(a) \otimes g(b) \end{aligned}$$

<u>Lemma</u> $(f_2 \otimes g_2)(f_1 \otimes g_1) = (f_2 f_1) \otimes (g_2 g_1)$

<u>Lemma</u> f, g isos $\Rightarrow$ $f \otimes g$ iso

<u>theorem</u>

$M \otimes_R - : R\text{-mod} \longrightarrow R\text{-mod}$ and $- \otimes_R M : R\text{-mod} \longrightarrow R\text{-mod}$

$$\begin{array}{ccc} A & & M \otimes_R A \\ f \downarrow & \longmapsto & \downarrow 1 \otimes f \\ B & & M \otimes_R B \end{array} \qquad \begin{array}{ccc} A & & A \otimes_R M \\ f \downarrow & \longmapsto & \downarrow f \otimes 1 \\ B & & B \otimes_R M \end{array}$$

are additive covariant functors.

Lemma there is a natural isomorphism $M \otimes_R N \cong N \otimes_R M$

Lemma $(A \otimes_R B) \otimes_R C \cong A \otimes_R (B \otimes_R C)$

thm there is a natural isomorphism

$$M \otimes_R \left(\bigoplus_i N_i\right) \xrightarrow{\cong} \bigoplus_{i \in I} (M \otimes_R N_i)$$

on both M and the family $\{N_i\}_{i \in I}$.

Lemma there is a natural isomorphism between

$R \otimes_R -$ and $Id : R\text{-mod} \longrightarrow R\text{-mod}$.

Sketch
$$\begin{aligned} R \times M &\longrightarrow M \\ (r, m) &\longmapsto rm \end{aligned}$$
is R-bilinear and surjective

$\Rightarrow$ induces homomorphism $R \otimes_R M \xrightarrow{\varphi_M} M$

$$\begin{aligned} M &\xrightarrow{f_M} R \otimes_R M \\ m &\longmapsto 1 \otimes m \end{aligned}$$
is an inverse to φ_M

the diagram
$$\begin{array}{ccc} R \otimes_R M & \xrightarrow{\varphi_M} & M \\ {\scriptstyle 1 \otimes g} \downarrow & & \downarrow {\scriptstyle g} \\ R \otimes_R N & \xrightarrow[\varphi_N]{} & N \end{array}$$
commutes

$\blacksquare$

Examples

① $\mathbb{Z}/2 \otimes_{\mathbb{Z}} \mathbb{Q} = 0$

$$a \otimes p = a \otimes \frac{2p}{2} = \underbrace{(2a)}_{=0} \otimes \frac{p}{2} = 0$$

all simple tensor $0 \Rightarrow$ tensor product is 0

② $\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} = 0$ because all simple tensors are 0:

$$\left(\frac{p}{q} + \mathbb{Z}\right) \otimes \left(\frac{a}{b} + \mathbb{Z}\right)$$

$$= \left(\frac{bp}{bq} + \mathbb{Z}\right) \otimes \left(\frac{a}{b} + \mathbb{Z}\right)$$

$$= \left(\frac{p}{bq} + \mathbb{Z}\right) \otimes b\left(\frac{a}{b} + \mathbb{Z}\right)$$

$$= \left(\frac{p}{bq} + \mathbb{Z}\right) \otimes \underbrace{(a + \mathbb{Z})}_{=0} = 0$$

③ $2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \neq 0$ because

$$2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z}$$

$$(a, b) \longmapsto \frac{ab}{2}$$

is $\mathbb{Z}$-bilinear, so it induces an R-module homomorphism

$$2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \xrightarrow{f} \mathbb{Z}/2\mathbb{Z}$$

and $f(2 \otimes 1) = \frac{2 \cdot 1}{1} = 1 \neq 0$

theorem $M\otimes_R -$ is right exact, meaning that for every ses

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

we get an exact sequence

$$M\otimes_R A \xrightarrow{1\otimes i} M\otimes_R B \xrightarrow{1\otimes p} M\otimes_R C \longrightarrow 0$$

Proof ① $(1\otimes p)(1\otimes i) = 0$

$$(1\otimes p)(1\otimes i) = 1\otimes \underbrace{pi}_{=0} = 0$$

② $1\otimes p$ is surjective

Given any $m_1\otimes c_1 + \cdots + m_n\otimes c_n \in M\otimes_R C$,

pick $b_1, \dots, b_n \in B$ such that $p(b_i) = c_i$ (p surjective)

then $(1\otimes p)\left(m_1\otimes b_1 + \cdots + m_n\otimes b_n\right) = m_1\otimes c_1 + \cdots + m_n\otimes c_n$.

③ $\ker(1\otimes p) = \operatorname{im}(1\otimes p)$

Let $I := \operatorname{im}(1\otimes p) \subseteq \ker(1\otimes p)$.

$1\otimes p$ induces a map $\quad M\otimes_R B/I \xrightarrow{q} M\otimes_R C$

Consider the canonical projection $\pi : M\otimes_R B \longrightarrow M\otimes_R B/I$

By construction: $q\pi = 1\otimes p$

$$\begin{array}{ccc} M \times C & \xrightarrow{f} & M \otimes_R B / I \\ (m, c) & \longmapsto & m \otimes b \end{array} \qquad \text{where } p(b) = c$$

- f is R-bilinear (exercise)
- f is well-defined:

if $p(b) = p(b') = c$, then

$b - b' \in \ker p = \operatorname{im} i \Rightarrow m \otimes (b - b') \in \operatorname{im}(1 \otimes i) = I$

so $\quad m \otimes b - m \otimes b' \in I$

$\Rightarrow$ f induces R-module map $\quad M \otimes_R C \xrightarrow{\hat{f}} M \otimes_R B / I$

<u>Claim</u> $\quad \hat{f} q = id_{M \otimes_R B / I}$

$$\hat{f} q \left(\sum_{i=1}^{n} m_i \otimes b_i \right) = \hat{f} \left(\sum_{i=1}^{n} m_i \otimes p(b_i) \right) = \sum_{i=1}^{n} m_i \otimes b_i$$

<u>Conclusion</u>: q is injective

$$q \pi = 1 \otimes p$$

$$\Rightarrow \ker(1 \otimes p) = \ker(q \pi) \overset{q \text{ injective}}{=} \ker \pi = \operatorname{im}(1 \otimes i)$$

But tensor is not necessarily exact

<u>Example</u> $0 \to \mathbb{Z} \xrightarrow{i} \mathbb{Q} \xrightarrow{\pi} \mathbb{Q}/\mathbb{Z} \to 0$

Apply $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} -$:

$$\underbrace{\mathbb{Z}/2 \otimes_{\mathbb{Z}} \mathbb{Z}}_{\cong \mathbb{Z}/2} \xrightarrow{1 \otimes i} \underbrace{\mathbb{Z}/2 \otimes_{\mathbb{Z}} \mathbb{Q}}_{=0} \longrightarrow \mathbb{Z}/2 \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} \to 0$$

<u>Hom-tensor adjunction</u> there is a natural isomorphism

$$\mathrm{Hom}_R(M \otimes_R N, P) \cong \mathrm{Hom}_R(M, \mathrm{Hom}_R(N, P))$$

$$f \longmapsto (m \longmapsto (n \longmapsto f(m \otimes n)))$$

$$m \otimes n \longmapsto g(m)(n) \longleftarrow\!\shortmid \quad g$$

What does this mean?
It says $(- \otimes_R M, \mathrm{Hom}_R(M, -))$ is an adjoint pair.

two covariant functors $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ form an <u>adjoint pair</u> if

$\mathrm{Hom}_{\mathcal{D}}(F(C), D) \cong \mathrm{Hom}_{\mathcal{C}}(C, G(D))$ naturally on C, D.

we say F is a <u>left adjoint</u> of G, G is a <u>right adjoint</u> of F

"the slogan is "adjoint functors arise everywhere"" Mac Lane

<u>Idea</u> An adjoint functor is a way of giving the most efficent solution to a problem in a functorial way

<u>Example</u>

<u>Problem</u> Given a set I, what is the most efficent functorial way of getting an R-module out of I?

<u>Solution</u> take the free R-module on I, $\bigoplus_I R$.

<u>Fact</u> $R\text{-mod} \underset{\text{Free}}{\overset{\text{Forget}}{\rightleftarrows}} \text{Set}$

(Free, Forget) form an adjoint pair.