One more thing about tensors:

thm $R \xrightarrow{f} S$ ring homomorphism

$\Big($ $\Leftrightarrow$ S is an R-algebra

$\Leftrightarrow$ S is a ring and an R-module via $r \cdot s := f(r) \cdot s$ $\Big)$

$S \otimes_R - : R\text{-mod} \longrightarrow S\text{-mod}$ is an additive functor.

Def the functor above is called <u>Extension of scalars</u>

$$f_! : R\text{-mod} \longrightarrow S\text{-mod}$$

Def the restriction of scalars functor

$$f^* : S\text{-mod} \longrightarrow R\text{-mod}$$

$M \longmapsto$ view M as an R-mod $r \cdot M := f(r) \cdot M$

$M \xrightarrow{g} N \longmapsto$ view g as a homomorphism of R-modules

<u>**theorem**</u> (Extension of scalars, Restriction of scalars) form an adjoint pair

More precisely:

<u>theorem</u>

If M is an R-module, and N is an S-module,

$$\mathrm{Hom}_R(M, f^* N) \cong \mathrm{Hom}_S(M \otimes_R S, N)$$

$$\mathrm{Hom}_R(M, \mathrm{Hom}_S(S, N))$$

<u>Remark</u> $\mathrm{Hom}_S(S, N)$ is an R-module by

$$r \cdot f = (s \longmapsto f(rs))$$

<u>Idea</u> Restriction of scalars solves the problem

Make an R-module out of a given S-module

Extension of scalars solves the problem

Make an S-module out of a given R-module

Previously, on <u>Homological Algebra</u>:

- $\mathrm{Hom}_R(M, -)$ is a left exact covariant functor
- $\mathrm{Hom}_R(-, M)$ is a left exact contravariant functor
- $M \otimes_R - \cong - \otimes_R M$ is a right exact covariant functor

<u>Long Term Goal</u> Measure the failure of these functors to be exact

<u>Short term goal</u> when are they exact?

<u>Def</u> An R-module P is projective if

$$\begin{tikzcd} & P \arrow[dl, dashed, "g"'] \arrow[d, "f"] & \\ A \arrow[r, "p"'] & B \arrow[r] & 0 \end{tikzcd}$$

For every surjective map $A \to B$ and every map $P \xrightarrow{f} B$ there exists $g : P \longrightarrow A$ making the diagram commute

so this says $p_*(g) = p \circ g = f$

we say g is a <u>lifting</u> of f

thm Free $\Rightarrow$ projective

Proof

$$\begin{tikzcd} & F \arrow[dl, dashed] \arrow[d, "f"] & \\ A \arrow[r, "p"'] & B \arrow[r] & 0 \end{tikzcd}$$

Let $\{b_i\}_{i \in I}$ be a basis for F.

$p(a_i) = b_i$ for some $a_i \in A$

F free $\Rightarrow$ there exists R-module map $g : F \to A$ induced by

$$g(b_i) := a_i$$

$\boxtimes$

Corollary For every R-module M, there exists $P \twoheadrightarrow M$ with P projective (free)

Proof $M = \sum_{i \in I} R m_i \Longrightarrow \bigoplus_{i \in I} R \twoheadrightarrow M$

$$(r_i) \longmapsto \sum_{i \in I} r_i m_i$$

thm $\mathrm{Hom}_R(P, -)$ exact $\Leftrightarrow$ P projective

Proof $\mathrm{Hom}_R(P)$ is left exact always, so

$0 \to A \to B \to C \to 0$ ses

$\Rightarrow 0 \to \mathrm{Hom}_R(P, A) \to \mathrm{Hom}_R(P, B) \to \mathrm{Hom}_R(P, C)$ exact

$\mathrm{Hom}_R(P, -)$ is exact if and only if

$B \xrightarrow{p} C \to 0 \Longrightarrow \mathrm{Hom}_R(B, P) \xrightarrow{g_*} \mathrm{Hom}_R(C, P) \to 0$

$\Leftrightarrow$ every $C \xrightarrow{f} P$ lifts to $B \xrightarrow{g} P$

$0 \to A \to B \to C \to 0$ is a <u>split</u> short exact sequence if it is isomorphic to

$$0 \to A \xrightarrow[i]{} A \oplus C \xrightarrow[p]{} C \to 0$$

where
- i is the inclusion of the first factor
- p is the projection onto the second factor

<u>Splitting lemma</u> $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ ses of R-mods

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] & B \arrow[r, "g"] \arrow[l, bend left, dashed, "q"] & C \arrow[r] \arrow[l, bend left, dashed, "r"] & 0 \end{tikzcd}$$

TFAE:

① there exists $q : B \to A$ such that $qf = id_A$

② There exists $r : C \to B$ such that $gr = id_C$

③ the short exact sequence splits

(r, q are called splittings for our ses)

<u>Proof</u> ③ $\Rightarrow$ ①, ②

If the sequence splits, say

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[r, "g"] \arrow[d, "b"'] & C \arrow[r] \arrow[d, "c"'] & 0 \\ 0 \arrow[r] & A \arrow[r, "i"] & A \oplus C \arrow[r, "p"] \arrow[l, bend left, dashed, "\pi_A"] & C \arrow[r] \arrow[l, bend left, dashed, "i_C"] & 0 \end{tikzcd}$$

isos

take

$q := a^{-1} \pi_A b$

$r := b^{-1} i_C c$

① $\Rightarrow$ ③

$$
\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] & B \arrow[r, "g"] \arrow[l, bend left, dashed, "q"] & C \arrow[r] & 0
\end{tikzcd}
\qquad qf = id_A
$$

Every $b \in B$ can be written as

$$b = \underbrace{(b - fq(b))}_{\ker q} + \underbrace{fq(B)}_{\in \operatorname{im} f \cong A}$$

$$\Uparrow$$

$$\Big(q(b) - \underbrace{qf}_{1}\, q(b) = 0\Big)$$

so $B = \ker q + \operatorname{im} f$

Also, if $f(a) \in \ker q$ then $a = qf(a) = 0$.

$\Rightarrow B = \ker q \oplus \operatorname{im} f \cong \ker q \oplus A$

<u>Claim</u> $\ker q \cong C$

$\ker q \cap \ker g = \ker q \cap \operatorname{im} f = 0$

$$\Longrightarrow \quad \begin{matrix} B & \xrightarrow{g} & C \\ \cup\!\vert & & \\ \ker q & \xrightarrow{g} & C \end{matrix} \quad \text{is injective}$$

It is also surjective:

Given $c \in C$, $g(b) = c$ for some $b \in B$, but $b = f(a) + \underset{\ker q}{\overset{\cap}{k}}$

$\Rightarrow c = g(b) = \underbrace{gf(a)}_{=0} + g(k) = g(k)$

So $\varphi : B \xrightarrow{\cong} \operatorname{im} f \oplus \ker g \xrightarrow{\cong} A \oplus C$ is an iso

$b \longmapsto (fq(b), b - fq(b)) \longmapsto (q(b), g(b))$

Now

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r, "g"] \arrow[d, "\varphi"] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A \arrow[r, "i"'] & A \oplus C \arrow[r, "p"'] & C \arrow[r] & 0
\end{tikzcd}$$

is an iso

- $\varphi f(a) = (\underbrace{qf}_{id}(a), \underbrace{gf}_{=0}(a)) = (a, 0) = i(a)$ ✓
- $p\varphi(b) = p(q(b), \underbrace{g(b)}) = g(b)$ ✓

② $\Rightarrow$ ③ Every $b \in B$ can be written as

$$b = \underbrace{(b - rg(b))}_{\in \ker g} - \underbrace{rg(b)}_{\in \operatorname{im} r}$$

$\Uparrow$

$$g(b) - \underbrace{gr}_{1} g(b) = 0$$

$\Rightarrow B = \ker g + \operatorname{im} r$

If $r(c) \in \ker g$, then $c = \underbrace{gr}_{=1}(c) = 0$.

$\Rightarrow B = \ker g \oplus \operatorname{im} r \cong A \oplus \operatorname{im} r$

Also : r injective $\Rightarrow$ $\operatorname{im} r \cong C$

$$\left(r(c) = 0 \Rightarrow c = gr(c) = 0 \right)$$

so $A \oplus C \xrightarrow[\psi]{\cong} B$

$$(a, c) \longmapsto f(a) + r(c)$$

so

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "i"] \arrow[d, equal] & A \oplus C \arrow[r, "p"] \arrow[d, "\psi"'] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A \arrow[r, "f"'] & B \arrow[r, "g"'] & C \arrow[r] & 0
\end{tikzcd}$$

- $\psi i(a) = \psi(i(a), 0) = f(i(a))$ ✓
- $g\psi(a,c) = g(f(a) + r(c)) = \underbrace{gf}_{=0}(a) + \underbrace{gr}_{1}(c) = c$ ✓

■

Not every ses splits!

Ex : $0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}/2 \to 0$ is not split!

$$\underbrace{\mathbb{Z}}_{\text{no 2-torsion}} \not\cong \underbrace{\mathbb{Z} \oplus \mathbb{Z}/2}_{\text{has 2-torsion}}$$

Warning Not every ses

$$0 \to A \to A \oplus C \to C \to 0 \quad \text{splits}$$

Example
$$0 \to \mathbb{Z}/2 \xrightarrow{f} \mathbb{Z}/(4) \xrightarrow{g} \mathbb{Z}/(2) \to 0$$
$$1 \longmapsto 2$$
$$1 \longmapsto 1$$

is not split! $\mathbb{Z}/(4) \not\cong \mathbb{Z}/(2) \oplus \mathbb{Z}/(2)$

$$M = \bigoplus_{\mathbb{N}} \left(\mathbb{Z}/(2) \oplus \mathbb{Z}/(4) \right)$$

so $\mathbb{Z}/(2) \oplus M \cong M \cong \mathbb{Z}/(4) \oplus M$

$$0 \longrightarrow \mathbb{Z}/(2) \xrightarrow{(f,0)} \mathbb{Z}/(4) \oplus M \xrightarrow{(g,\, id)} \mathbb{Z}/(2) \oplus M \longrightarrow 0$$
$$q$$

this is not split: any splitting q would restrict to $\mathbb{Z}/(4) \longrightarrow \mathbb{Z}/(2)$ that splits the original sequence

Warning the splitting lemma is false in Grp.

Example $$0 \longrightarrow A_3 \underset{\text{inclusion}}{\xrightarrow{i}} S_3 \xrightarrow{\text{sign}} \mathbb{Z}/2 \longrightarrow 0$$

- this is not split: $\underset{\text{non-abelian}}{S_3} \not\cong \underset{\text{abelian}}{A_3 \oplus \mathbb{Z}/2}$

- but $u : \mathbb{Z}/2 \longrightarrow S_3$ satisfies $\text{sign} \circ u = id_{\mathbb{Z}/2}$

  $1 \longmapsto$ any 2-cycle