Previously, on Homological Algebra:

- $\mathrm{Hom}_R(M,-) : R\text{-mod} \to R\text{-mod}$ is left exact
- $\mathrm{Hom}_R(-,M) : R\text{-mod} \to R\text{-mod}$ is left exact
- $M \otimes_R - : R\text{-mod} \to R\text{-mod}$ is right exact

$\mathrm{Hom}_R(P,-)$ is exact $\Leftrightarrow$ P is <u>projective</u> $\Leftrightarrow$

$$\begin{tikzcd} & P \arrow[dl, dashed, "g"'] \arrow[d, "f"] & \\ A \arrow[r] & B \arrow[r] & 0 \end{tikzcd}$$

Free $\Rightarrow$ projective

<u>thm</u> For every module M, there exists projective P such that $P \twoheadrightarrow M$.

A ses is <u>split</u> if it is isomorphic to

$$0 \to A \xrightarrow{i} A \oplus C \xrightarrow{\pi} C \to 0$$

inclusion of 1^{st} factor   projection onto 2^{nd} factor

$$\Leftrightarrow \quad \begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] & B \arrow[r, "g"] \arrow[l, dashed, bend left, "q"] & C \arrow[r] & 0 \end{tikzcd} \qquad qf = id_A$$

$$\Leftrightarrow \quad \begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] & B \arrow[r, "g"] & C \arrow[r] \arrow[l, dashed, bend left, "r"] & 0 \end{tikzcd} \qquad gr = id_C$$

thm P is projective $\iff$ every $0 \to A \to B \to P \to 0$ splits.

Proof $0 \to A \xrightarrow{f} B \xrightarrow{g} P \to 0$ ses

P projective $\Rightarrow$
$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists h"'] \arrow[d, equal] & \\ B \arrow[r, "g"'] & P \arrow[r] & 0 \end{tikzcd}$$
$\Rightarrow$ h is a splitting

Suppose every ses $0 \to A \to B \to P \to 0$ splits

Given
$$\begin{tikzcd} & P \arrow[d, "f"] & \\ B \arrow[r, "p"'] & C \arrow[r] & 0 \end{tikzcd}$$

free
$F \overset{\pi}{\twoheadrightarrow} P \rightsquigarrow$ $0 \to \ker \pi \to F \xrightarrow{\pi} P \to 0$ splits
$$\begin{tikzcd} F \arrow[r, "\pi"] & P \arrow[l, bend left, dashed, "h"] \end{tikzcd}$$

F free $\Rightarrow \exists$ R-mod map $F \xrightarrow{\hat{g}} B$ such that

$$\begin{tikzcd} F \arrow[r, "\pi"'] \arrow[d, dashed, "\hat{g}"'] & P \arrow[l, bend right, dashed, "h"'] \arrow[d, "f"] \arrow[dl, dotted, "g"] & \\ B \arrow[r, "p"'] & C \arrow[r] & 0 \end{tikzcd}$$

set $g := \hat{g} h$. then $pg = p\hat{g}h = f \underbrace{\pi h}_{1} = f$. $\square$

An R-module M is a direct **summand** of N if $A \oplus M \cong N$ for some A.

thm P is projective $\Leftrightarrow$ P is a direct summand of a free module.

If P is fg, P projective $\Leftrightarrow$ P direct summand of R^n

Proof $(\Rightarrow)$ P projective, $\overset{\text{free}}{F} \xrightarrow{\pi} P$ ($F = R^n$ if fg)

$$0 \longrightarrow \ker\pi \longrightarrow F \xrightarrow{\pi} P \longrightarrow 0 \quad \text{ses} \Rightarrow \text{splits}$$

$\therefore$ $F \cong P \oplus \ker \pi$:)

$(\Leftarrow)$ P direct summand of the free module F

$\pi : F \longrightarrow P$, $i : P \to F$ $\quad \pi i = id_P$
projection inclusion

$$\begin{tikzcd} F \arrow[r, "\pi"'] \arrow[d, dotted, "h"'] & P \arrow[l, bend right, "i"'] \arrow[d, "f"] \\ B \arrow[r, "p"'] & C \arrow[r] & 0 \end{tikzcd}$$

$g := hi$ is a lifting of f

$$pg = phi \underset{ph = f\pi}{=} f\pi i \underset{\pi i = id_P}{=} f$$

$\square$

Corollary

- Every direct summand of a projective module is projective
- Every direct sum of projectives is projective.

Projective $\not\Rightarrow$ free

Example $R = \mathbb{Z}/6 = \underbrace{(2)}_{I} \oplus \underbrace{(3)}_{J} \Rightarrow I, J$ projective

fg free modules have $6n$ elements $\Rightarrow I, J$ not free

thm (R, m) local ring. Every fg projective R-module is free

Slogan free $\Leftrightarrow$ projective over a local ring

Proof P fg projective, $n = \mu(P)$ minimal generating set

$$F = R^n \xrightarrow{\pi} P = Rm_1 + \cdots + Rm_n$$
$$(r_1, \ldots, r_n) \longmapsto r_1 m_1 + \cdots + r_n m_n$$

$\{m_1, \ldots, m_n\}$ minimal generating set $\Leftrightarrow$ basis for P/mP

$(r_1, \ldots, r_n) \in \ker \pi \Rightarrow r_1, \ldots, r_n \in m$

so $\ker \pi \subseteq mF$

P projective $\Rightarrow$ $0 \to \ker\pi \to F \xrightarrow{\pi} P \to 0$ splits, with $j : P \to F$

$F \cong \operatorname{im} j \oplus \ker \pi$

$\ker \pi \subseteq mF \Rightarrow \ker\pi \subseteq m(\ker \pi) \overset{\text{NAK}}{\Rightarrow} \ker \pi = 0$

$\therefore$ π is an iso and P is projective.

Injectives

I is injective $\iff$
$$\begin{tikzcd} & I & \\ 0 \arrow[r] & A \arrow[u, "f"] \arrow[r] & B \arrow[ul, dashed, "g"'] \end{tikzcd}$$

lemma I injective $\iff \mathrm{Hom}_R(-, I)$ is exact.

Proof $\mathrm{Hom}_R(-, I)$ left exact

$\mathrm{Hom}_R(-, I)$ exact $\iff$ for every $0 \to A \xrightarrow{i} B$, i^* is surjective

$\iff$ for every f there exits g
$$\begin{array}{ccc} \mathrm{Hom}_R(B, I) & \xrightarrow{i^*} & \mathrm{Hom}_R(A, I) \\ g & \mapsto & f \end{array}$$

$\square$

lemma I_i injective for all $i \Rightarrow \prod_i I_i$ injective

Idea: to map into a product is to map into each factor.

thm If $M \oplus N = E$ is injective $\Rightarrow M, N$ are injective

Sketch
$$\begin{tikzcd} & M \arrow[r] & E \arrow[l, bend right] \\ 0 \arrow[r] & A \arrow[u] \arrow[r] & B \arrow[u, dotted] \end{tikzcd}$$

Baer Criterion E is injective if and only if for every ideal I, every R-module map $I \to E$ can be extended to R

$$\begin{tikzcd} & E & \\ 0 \arrow[r] & I \arrow[u, "f"] \arrow[r] & R \arrow[ul, dashed, "g"'] \end{tikzcd}$$

Proof this is a special case of the definition, so we only need to show this condition implies injectivity

$$\begin{tikzcd} & E & \\ 0 \arrow[r] & M \arrow[u, "f"] \arrow[r] & N \end{tikzcd} \qquad \text{Assume } M \subseteq N$$

$X := \{ (A, g) \mid M \subseteq A \subseteq N \text{ submodule}, \ g \text{ extends } f \}$

$\neq \emptyset$ because $(M, f) \in X$

X is partially ordered by

$(A, g) \leq (B, h) := A \subseteq B$ and $h|_A = g$

Claim Can apply Zorn's lemma to X.

Recall Zorn's lemma If S is a partially ordered set and every chain in S has an upper bound, then S has a maximal element.

Why we can apply Zorn's Lemma:

$$(A_1, g_1) \leq (A_2, g_2) \leq \cdots$$

$$A := \bigcup_{i=1}^{\infty} A_i \xrightarrow{g} E$$
$$a \longmapsto g_i(a) \quad \text{if } a \in A_i$$

thus (A, g) is an upper bound for our chain.

$\therefore$ By Zorn's Lemma, X has a maximal element, say (A, g).

<u>Claim</u> $A = N$.

Suppose $n \in N$, $n \notin A$.

$$I := \{ r \in R \mid rn \in A \} \quad \text{is an ideal}$$

$$I \xrightarrow{h} E \qquad \text{is an } R\text{-module map}$$
$$r \longmapsto g(rn)$$

By assumption, this h extends to $R \xrightarrow{h} E$.

$$A + Rn \xrightarrow{\varphi} E \qquad \text{is an } R\text{-mod map}$$
$$a + rn \longmapsto g(a) + h(r)$$

is well-defined: if $rn \in A$, $h(r) = g(rn)$ ✓
and it extends g! so $A = N$ by maximality $\lightning$ ▢

Corollary R Noetherian

$\{M_i\}_i$ injective $\Rightarrow$ $\bigoplus_i M_i$ is injective

Proof

$$\begin{tikzcd} & \bigoplus M_i & \\ 0 \arrow[r] & I \arrow[u] \arrow[r] & R \end{tikzcd}$$

R Noetherian $\Rightarrow$ $I = (a_1, \dots, a_n)$

$f(a_i) = (b_{ij})_j$ $\quad$ $b_{ij} \neq 0$ only for finitely many j

$k := \{ j \mid f(a_i)_j = b_{ij} \neq 0 \ \text{ for some } i \}$

$f(I) \subseteq \bigoplus_{j \in k} M_j = \prod_{j \in k} M_j$ injective

finite!

so f extends to $R \longrightarrow \bigoplus_{j \in k} M_j \hookrightarrow \bigoplus_j M_j$ $\boxtimes$