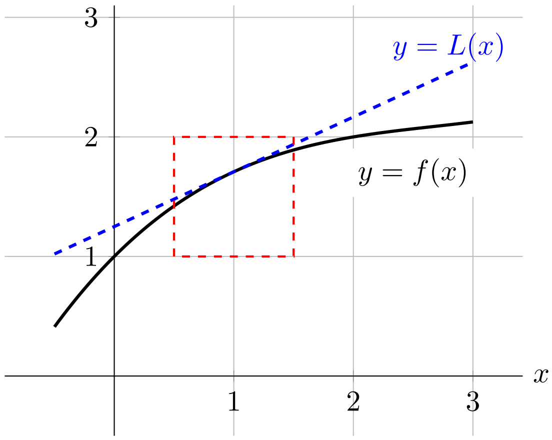
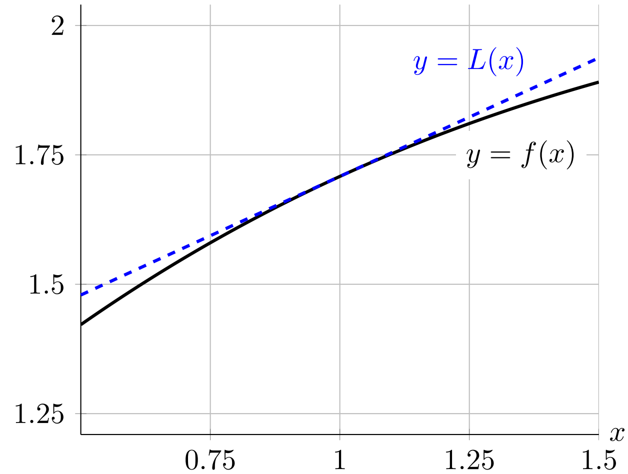
Math 115: Linear and quadratic approximations of differentiable functions

In Section 3.9, you learned that a differentiable function $f(x)$ looks like a straight line $L(x)$ when you zoom in on its graph near a point. The graphs below show the graph of $y = L(x)$ near $x = 1$.





The linear function $L(x)$ near the point $x = a$ is given by the formula

$$L(x) = f(a) + f'(a)(x - a).$$

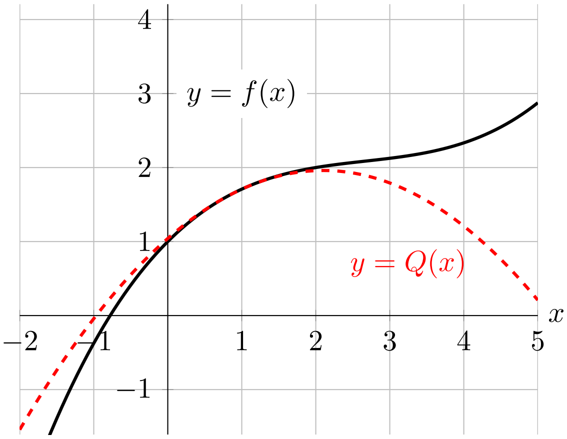
In the graph above, the value of $a = 1$. Notice that the graph of $y = L(x)$ is only close to the graph of $y = f(x)$ for values of x close to 1. We call $L(x)$ the tangent line approximation (or local linearization) of $f(x)$ near $x = a$. Notice that the value of $x = a$, the function $L(x)$ has the same output value and derivative value as $f(x)$.

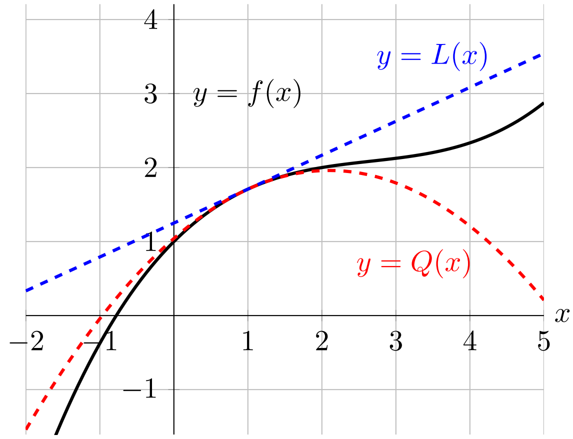
One reason why the linear approximations are widely used to approximate values of differentiable functions is because the formulas for the linear function $L(x)$ is simple. However, this simplicity can mean that the linear approximation may not be a very close approximation for $f(x)$. We may want another function that is still fairly simple, but approximates $f(x)$ more closely. If the function $f(x)$ is twice differentiable, then its values can be approximated using quadratic functions. Suppose we want to approximate values of a differentiable function $f(x)$ near the point $x = a$ with a quadratic function

$$Q(x) = A + B(x - a) + C(x - a)^2$$

where A,B and C are constants whose values need to be determined based on the values of the function $f(x)$ near $x = a$. We use $(x - a)$ and $(x - a)^2$ instead of x and x^2 because this makes it easier to think about the properties of the approximation at $x = a$. Just as with the linear approximation, we want $Q(x)$ to have the same value and slope at $x = a$ as $f(x)$, but now we can also require that the second derivatives be equal. That is, we want to find A, B and C such that $Q(a) = f(a)$, $Q'(a) = f'(a)$ and $Q''(a) = f''(a)$. In order for this to be true $A = f(a)$, $B = f'(a)$ and $C = \frac{1}{2}f''(a)$. (You should take the time to find $Q'(a)$ and $Q''(a)$ to verify this for yourself.) Hence we get the formula

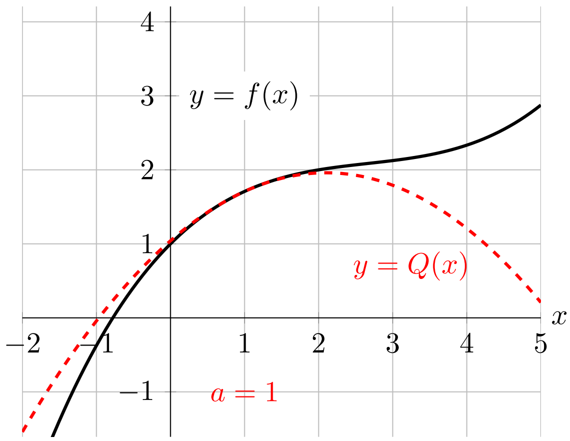
$$Q(x) = f(a) + f'(a)(x - a) + \frac{1}{2}f''(a)(x - a)^2.$$

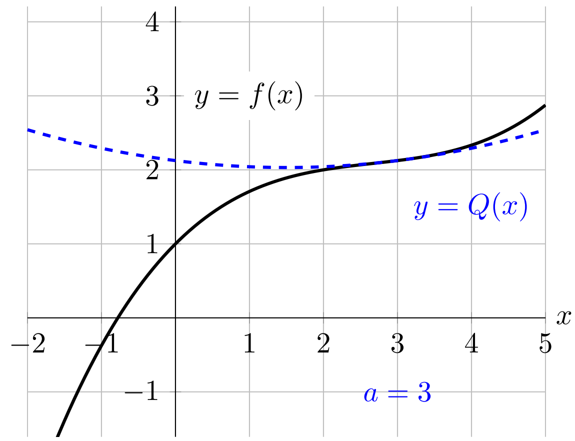




There are a couple of things that are important to keep in mind regarding the formula for $Q(x)$:

- The formula of $Q(x)$ depends in the values of a. This means that this formula will give us different parabolas that approximate the values of $f(x)$ at different points.





- The first two terms in the formula of $Q(x)$ correspond to the terms in the formula for the tangent line approximation $L(x)$ of $f(x)$. In a sense, the new term in the formula of $Q(x)$ is a term that bends the graph of the tangent lines closer to the graph of $f(x)$.

In order to understand this formula, we will work a couple of examples:

Example 1: Find the linear and quadratic approximations to the following functions near the indicated points.

i) $f(x) = \dfrac{2}{x+1}$ near $x = 2.2$.

ii) $g(x) = xe^{-2x}$ near $x = 1$.

iii) $h(p) = \sqrt{1-4p}$ near $p = -2$.

iv) $j(z) = \ln(bz^2 + 6)$ near $z = 0$ where b is a positive constant.

Solution:

i) $f(x) = \dfrac{2}{x+1}$ near $x = 2.2$.
Since

$$f'(x) = -2(x+1)^{-2}$$
$$f''(x) = 4(x+1)^{-3}$$

then

$$L(x) = f(2.2) + f'(2.2)(x-2.2) = \frac{1}{32} - \frac{1}{512}(x-2.2)$$

and

$$Q(x) = f(2.2) + f'(2.2)(x-2.2) + \frac{1}{2}f''(2.2)(x-2.2)^2$$
$$= \frac{1}{32} - \frac{1}{512}(x-2.2) + \frac{1}{16384}(x-2.2)^2.$$

ii) $g(x) = xe^{-2x}$ near $x = 1$.
Since

$$g'(x) = e^{-2x} - 2xe^{-2x} = e^{-2x}(1-2x)$$
$$g''(x) = -2e^{-2x} - 2(e^{-2x} - 2xe^{-2x}) = -4e^{-2x} + 4xe^{-2x} = -4e^{-2x}(1-x).$$

then

$$L(x) = g(1) + g'(1)(x-1) = e^{-2} - e^{-2}(x-1)$$

and

$$Q(x) = e^{-2} - e^{-2}(x-1) = L(x).$$

iii) $h(p) = \sqrt{1-4p}$ near $p = -2$.
Since

$$h'(p) = \frac{1}{2}(1-4p)^{-\frac{1}{2}}(-4) = -2(1-4p)^{-\frac{1}{2}}$$
$$h''(p) = -\frac{1}{2}(-2)(1-4p)^{-\frac{3}{2}}(-4) = -4(1-4p)^{-\frac{3}{2}}$$

then

$$L(p) = h(-2) + h'(-2)(p+2) = 3 - \frac{2}{3}(p+2)$$

and

$$Q(p) = h(-2) + h'(-2)(p+2) + \frac{1}{2}h''(-2)(p+2)^2$$
$$= 3 - \frac{2}{3}(p+2) - \frac{2}{27}(p+2)^2.$$

iv) $j(z) = \ln(bz^2 + 6)$ near $z = 0$ where b is a positive constant.
Since

$$j'(z) = \frac{1}{bz^2+6}(2bz) = \frac{2bz}{bz^2+6}$$

$$j''(z) = \frac{2b(bz^2+6) - 2bz(2bz)}{(bz^2+6)^2} = \frac{2b^2z^2 + 12b - 4b^2z^2}{(bz^2+6)^2} = \frac{12b - 2b^2z^2}{(bz^2+6)^2}$$

then

$$L(z) = j(0) + j'(0)z = \ln(6)$$

and

$$Q(z) = j(0) + j'(0)z + \frac{1}{2}j''(0)z^2 = \ln(6) + \frac{b}{6}z^2.$$

Example 2: Let $f(x) = 4 + \sin(2x)$.

i) Find the linear ($L(x)$) and quadratic ($Q(x)$) approximations to the function $f(x)$ near $x = 1$.
ii) Approximate the values of $f(x)$ for $x = 0.5, 0.75, 1, 1, 25$ and 1.5 using the functions $L(x)$ and $Q(x)$. Compare the values of your approximations to the actual values of $f(x)$ at these values. What do you notice?
iii) Use the graphs of the functions $f(x)$, $L(x)$ and $Q(x)$ to determine which of these functions approximate better the values of $f(x)$ for $0.5 \leq x \leq 1.5$.
iv) Let $P(x)$ be the quadratic approximation to $f(x)$ near $x = \frac{\pi}{2}$.
   a) Find a formula for $P(x)$.
   b) Without evaluating $P(x)$, which quadratic approximation to $f(x)$ would you expect to give you a closer value to $f(1.5)$, $P(x)$ or $Q(x)$? Justify.
   c) Find the values of $P(1.5)$ to verify your answer in b).

Solution: Using the formulas we obtain that the linear and quadratic approximation functions $L(x)$ and $Q(x)$ of $f(x)$ near $x = 1$ are given by

$$L(x) = f(1) + f'(1)(x-1) = 2 + \sin(2) + 2\cos(2)(x-1)$$

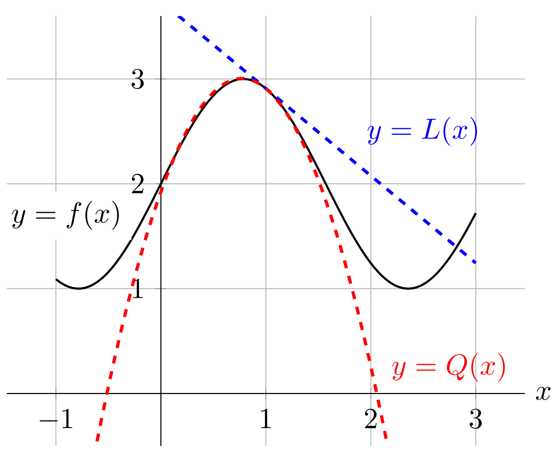
and

$$\begin{aligned} Q(x) &= f(1) + f'(1)(x-1) + \frac{1}{2}f''(1)(x-1)^2 \\ &= 2 + \sin(2) + 2\cos(2)(x-1) - 2\sin(2)(x-1)^2. \end{aligned}$$

Using these formulas, we compute their values at $x = 0.5, 0.75, 1, 1, 25$ and 1.5 accurate up to their first three decimals. Our results are recorded in the table below

x	0.5	0.75	1	1.25	1.5
$f(x)$	2.841	2.997	**2.909**	2.598	2.141
$L(x)$	3.254	3.117	**2.909**	2.701	2.493
$Q(x)$	2.870	3.003	**2.909**	2.587	2.038

Graphing all the functions we can observe that $Q(x)$ better approximates the values of $f(x)$ close to $x = 1$. In general, we expect the quadratic approximation $Q(x)$ to produce a better approximation to the values of $f(x)$ than the tangent line approximation for values close to $x = 1$. This is the reason why we are interested in quadratic approximations.



In part iv), you are going to use two quadratic approximations to $f(x)$ near two different points to make approximations to the value of $f(1.5)$. The quadratic approximation $Q(x)$ is centered at $a = 1$ while the other one, $P(x)$, is centered at $a = \frac{\pi}{2} \approx 1.5707$. Using the fact that

$$f\left(\frac{\pi}{3}\right) = 2 + \sin \pi = 2.$$
$$f'\left(\frac{\pi}{3}\right) = 2\cos(\pi) = -2.$$
$$f''\left(\frac{\pi}{3}\right) = -4\sin(\pi) = 0.$$

Then $P(x) = 2 - 2\left(x - \dfrac{\pi}{2}\right)$. Notice that in this case, the quadratic approximation of $f(x)$ near $x = \dfrac{\pi}{2}$ is equal to the linear approximation of $f(x)$ at that point.
Since $P(x)$ is centered at $x = \frac{\pi}{2}$, a point that is closer to $x = 1.5$ compared to $x = 1$, then we expect $P(1.5)$ to be a better approximation to $f(1.5)$.
Using our formula, we get $P(1.5) = 2 - 2\left(1.5 - \dfrac{\pi}{2}\right) \approx 2.1415$. From our previous computations we see that $P(1.5)$ is closer to $f(1.5)$ than $Q(1.5)$.