Worksheet 22

Problem 1 (Winter 2018 Exam 2 Problem 7)**.** The amount of chlorine in a chemical reaction C(t) (in gallons) t seconds after it has been added into a solution is given by the function

$$C(t) = 2 - 3(t-5)^{\frac{4}{5}}(t-1)e^{-t} \qquad \text{for } t \geqslant 0.$$

Notice that

$$C'(t) = \frac{3(t-6)(5t-9)e^{-t}}{5(t-5)^{\frac{1}{5}}} \qquad \text{for } t \geqslant 0.$$

(a) Use calculus to find the time(s) (if any) at which the amount of chlorine in the solution is the greatest and the smallest.

Problem 2 (Fall 2017 Exam 2 Problem 5)**.** Blizzard the snowman and his mouse friend Gabe arrived in Montana, where it has recently snowed. Since Blizzard is still melting, they decide to use this time to pack extra snow onto Blizzard, to help him make it to the North Pole. Let $H(t)$ be Blizzard's height, in inches, if Blizzard and Gabe stay in Montana for t hours. On the interval $1 \leqslant t < \infty$, the function $H(t)$ can be modeled by

$$H(t) = 35 + 10e^{-\frac{t}{6}}(t-2)^{\frac{1}{3}}.$$

Notice that

$$H'(t) = \frac{-5e^{-\frac{t}{6}}(t-4)}{(t-2)^{\frac{1}{3}}}.$$

(a) Find all values of t that give global extrema of the function $H(t)$ on the interval $1 \leqslant t < \infty$. Use calculus to find your answers, and be sure to show enough evidence that the point(s) you find are indeed global extrema. For each answer, write none if appropriate.

(b) Assuming Blizzard stays in Montana for at least 1 hour, what is the tallest height Blizzard can reach? Remember to include units.

Problem 3 (Winter 2017 Exam 2 Problem 8)**.** At Happy Hives Bee Farm, the population of bees, in thousands, t months after the farm opens, can be modeled by $g(t)$, where

$$g(t) = \begin{cases} 20 + \frac{1}{3}e^{4-t} & \text{for } 0 \leqslant t \leqslant 4 \\ -\frac{1}{6}t^3 + \frac{1}{4}t^2 - 7t + 23 & \text{for } 4 < t \leqslant 8 \end{cases}$$

and

$$g'(t) = \begin{cases} -\frac{1}{3}e^{4-t} & \text{for } 0 \leqslant t \leqslant 4 \\ -0.5(t-2)(t-7) & \text{for } 4 < t \leqslant 8 \end{cases}$$

(a) Find the values of t that minimize and maximize $g(t)$ on the interval $[0, 8]$. Use calculus to find your answers, and be sure to show enough evidence that the points you find are indeed global extrema. For each answer, write none if appropriate.

(b) What is the largest population of bees that occurs in the first 8 months the farm is open?

Problem 4. Circle the answer that correctly completes the statement.

(a) Suppose $f(x) = x^4 - 2a^2x^2 + 2a^2$ where $a > 2$ is a positive constant. The critical points of $f(x)$ are at $x = 0, \pm a$. Then the global maximum of $f(x)$ on $[-2a, 1]$ occurs at

$x = a$ $\quad$ $x = -a$ $\quad$ $x = \pm a$ $\quad$ $x = 0$ $\quad$ $x = -2a$ $\quad$ $x = 1$

(b) Suppose $f(x) = x^4 - 2a^2x^2 + 2a^2$ where $a > 2$ is a positive constant. The critical points of $f(x)$ are at $x = 0, \pm a$. Then the global minimum of $f(x)$ on $[-2a, 1]$ occurs at

$x = a$ $\quad$ $x = -a$ $\quad$ $x = \pm a$ $\quad$ $x = 0$ $\quad$ $x = -2a$ $\quad$ $x = 1$

Problem 5. Saruman is creating an orc army to cut down all trees in Fangorn forest. His research indicates that an army of x thousand orcs, will be able to cut down

$$T(x) = \frac{x^3}{3} - 3x^2 + 8x \qquad \text{thousand trees per hour.}$$

(i) If Saruman is capable of producing an army of up to 3000 orcs, how many should he produce in order to maximize the hourly destruction of trees? Saruman must be convinced by the methods of calculus.

(ii) Does your answer change if Saruman can produce up to 4000 orcs? If so, how many should he produce now?

(iii) Does your answer change if Saruman can produce up to 6000 orcs? If so, how many should he produce now?

Problem 6. Bill wants to minimize how long it will take him to complete his upcoming quiz grading. Before starting to grade, he buys a cup of coffee containing 55 miligrams of caffeine. Let $H(x)$ be the number of minutes it will take Bill to complete his grading if he consumes x miligrams of caffeine,

$$H(x) = \frac{x^2}{120} - \frac{4x}{3} + 2 + 20\ln(x).$$

Being very math-minded, instead of immediately starting to grade he decides to solve a calculus problem to determine how much caffeine he should consume.

(a) Find all the values of x at which $H(x)$ attains global extrema on the interval $10 \leqslant x \leqslant 55$. Use calculus to find your answers, and be sure to show enough evidence that the points you find are indeed global extrema.

(b) Assuming Bill consumes at least 10 miligrams of caffeine and at most 55 miligrams of caffeine, what is the shortest amount of time it could take for him to finish the grading? Remember to include units!

Problem 7 (Fall 2017 Exam 2 Problem 6)**.** Sketch graphs of functions $f(x)$ and $g(x)$ satisfying the conditions below, or say that no such function exists.

(a) A function $f(x)$ defined on the interval $(0,4)$ that satisfies

- $f'(x) > 0$ for all $x \neq 2$, and
- $x = 2$ is a global minimum.

(b) A function $g(x)$ defined on the interval $(0,4)$ that satisfies

- $\lim_{x \to 2^-} g'(x) = \infty$, and
- $\lim_{x \to 2^+} g'(x) = 0$.

Problem 8 (Winter 2017 Exam 2 Problem 7)**.** Sketch the graph of a single function $y = h(x)$ satisfying all the following:

- The function $h(x)$ is defined for $-7 \leqslant x \leqslant 7$.
- $h(x)$ has global maximums at $x = 7$ and $x = 3$.
- $h(x)$ has an inflection point at $x = -5$.
- $h(x)$ is continuous at $x = -3$ but not differentiable at $x = -3$.
- $h(x)$ has a local minimum at $(-1, -4)$ but is not continuous at $x = -1$.
- $h(x)$ has a critical point at $(2, 5)$ that is neither a local maximum or a local minimum.
- $h(x)$ satisfies the conclusion of the Mean Value Theorem on $[4, 7]$ but not the hypothesis of this theorem.

Problem 9 (Winter 2016 Exam 2 Problem 6)**.** Sketch the graph of a single function $y = g(x)$ satisfying all the following:

- $g(x)$ is defined for all x in the interval $-6 < x < 6$.
- $g(x)$ has at least 5 critical points in the interval $-6 < x < 6$.
- The global maximum value of $g(x)$ on the interval $-5 \leqslant x \leqslant -3$ is 4, and this occurs at $x = -4$.
- $g(x)$ is not continuous at $x = -2$.
- $g'(x)$ (the derivative of g) has a local maximum at $x = 0$.
- $g(x)$ is continuous but not differentiable at $x = 1$.
- $g''(x) \geqslant 0$ for all x in the interval $2 < x < 4$.
- $g(x)$ has at least one local minimum on the interval $4 < x < 6$ but does not have a global minimum on the interval $4 < x < 6$.
- $g(x)$ has an inflection point at $x = 5$.