Worksheet 24

Problem 1 (Winter 2017 Final Exam Problem 9). A Math 115 coordinator is trying to create functions with certain properties in order to test students' understanding of various concepts.

(a) She wants a function $f(x)$ of the form

$$f(x) = \begin{cases} ax^2 + ax + be^x & \text{for } x \leqslant 0 \\ a + 2\cos(x) & \text{for } x > 0 \end{cases}$$

with a and b constants. Find all value(s) of a and b for which $f(x)$ is differentiable at $x = 0$.

(b) The coordinator also wants a function $g(x) = cx - e^x$, where c is a constant, so that $g(x)$ has at least one critical point. What condition(s) on c will make this true? Find the x-values of all critical points in this case. Your answer may be in terms of c.

Problem 2 (Fall 2017 Final Exam Problem 10). Consider the family of functions $g(x) = e^x - kx$, where k is a positive constant.

(a) Show that the point $(\ln(k), k - k\ln(k))$ is the only critical point of $g(x)$ for all positive k. Show all your work to receive full credit.

(b) Show that $g(x)$ has a global minimum on $(-\infty, \infty)$ at $x = \ln(k)$.

(c) Find all values of $0.5 \leqslant k \leqslant 2$ that maximize the y-value of the global minimum of $g(x)$ on $(-\infty, \infty)$.

Problem 3 (Fall 2016 Final Exam Problem 12). Let W be the differentiable function given by

$$W(p) = \begin{cases} 4\ln(2) + 4\ln(-p) & \text{for } p \leqslant -0.5 \\ 2\sin(4p^2 - 1) & \text{for } -0.5 < p < 0.5 \\ \frac{\arctan(2p-1)}{p^2} & \text{for } p \geqslant 0.5 \end{cases}$$

(a) Use the limit definition of the derivative to write an explicit expression for $W'(3)$. Your answer should not involve the letter W. Do not evaluate or simplify the limit.

(b) With W as defined above, consider the function g defined by

$$g(t) = \begin{cases} ct + k & \text{for } t \leqslant 0 \\ W(-e^t) & \text{for } t > 0 \end{cases}$$

for some constants c and k. Find all values of c and k so that $g(t)$ is differentiable. Show your work carefully, and leave your answers in exact form.

Problem 4 (Winter 2017 Final Exam Problem 7). Consider the family of functions $f(x) = ax^2e^{-bx}$, where a and b are positive constants. Note that

$$f'(x) = ax(2 - bx)e^{-bx}.$$

(a) Find the exact values of a and b so that $f(x)$ has a critical point at $(4, e^{-2})$.

(b) Using your values of a and b from the previous part, find and classify the local extrema of $f(x)$.

Problem 5 (Winter 2015 Final Exam Problem 2)**.** For nonzero constants a and b with $b > 0$, consider the family of functions given by $f(x) = e^{ax} - bx$. Note that the derivative and second derivative of $f(x)$ are given by

$$f'(x) = ae^{ax} - b \text{ and } f''(x) = a^2 e^{ax}.$$

(a) Suppose the values of a and b are such that $f(x)$ has at least one critical point. For the domain $(-\infty, \infty)$, find all critical points of $f(x)$, all values of x at which $f(x)$ has a local extremum, and all values of x at which $f(x)$ has an inflection point. Use calculus to find and justify your answers, and be sure to show enough evidence to demonstrate that you have found all local extrema and inflection points.

(b) Which of the following conditions on the constant a guarantee(s) that $f(x)$ has at least one critical point in its domain $(-\infty, \infty)$? Circle all the cases in which $f(x)$ definitely has at least one critical point. Hint: There is at least one such condition listed.

$$a < 0 \qquad\qquad 0 < a < b \qquad\qquad b < a$$

(c) Find exact values of a and b so that $f(x)$ has a critical point at $(1, 0)$. Remember to show your work carefully.

Problem 6 (Fall 2014 Exam 3 problem 9)**.** Consider the family of functions given by

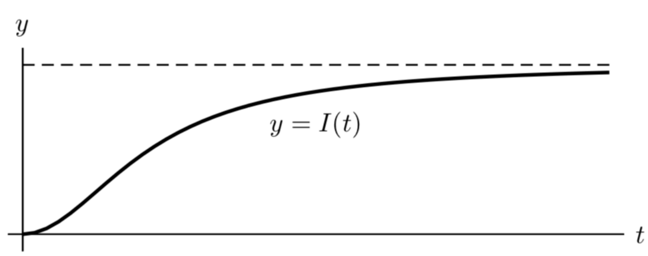
$$I(t) = \frac{At^2}{B + t^2}$$

where A and B are positive constants. Note that the first and second derivatives of $I(t)$ are

$$I'(t) = \frac{2ABt}{(B + t^2)^2} \qquad \text{and} \qquad I''(t) = \frac{2AB(B - 3t^2)}{(B + t^2)^3}.$$

(a) Find $\lim_{t\to\infty} I(t)$. Your answer may include the constants A and/or B.

(b) A researcher studying the ice cover over Lake Michigan throughout the winter proposes that for appropriate values of A and B, the function $I(t)$ is a good approximation for the number of thousands of square miles of Lake Michigan covered by ice t days after the start of December. For such values of A and B, a graph of $y = I(t)$ for $t \geqslant 0$ is shown below.



Based on observations, the researcher chooses values of the parameters A and B so that the following are true.

- $y = 21$ is a horizontal asymptote of the graph of $y = I(t)$.
- $I(t)$ is increasing the fastest when $t = 25$.

Find the values of A and B for the researcher's model.

Problem 7 (Winter 2013 Exam 2 problem 3)**.** Consider the family of linear functions

$$L(x) = ax3$$

and the family of functions

$$M(x) = a\sqrt{x}$$

where a is a nonzero constant number. Note that the number a is the same for both equations. Find a value of a for which $L(x)$ is tangent to the graph of $M(x)$. Also find the x and y coordinates of the point of tangency.

Problem 8 (Winter 2016 Final Exam problem 7)**.** Consider the family of functions $f(x) = e^{x^2+Ax+B}$ for constants A and B.

(a) Find and classify all local extrema of f.

(b) Find all the values of A and B so that the point $(3, 1)$ is a critical point of $f(x)$. Use calculus to find and justify your answers, and be sure to show enough evidence to demonstrate that you have found all local extrema. For each answer blank, write none if appropriate. Your answers may depend on A and/or B.

Problem 9 (Winter 2015 Exam 2 problem 7, adapted)**.** To aid in Elphaba's escape, Walt has concocted a supplement that will make her stronger and more agile. The concentration of the supplement in Elphaba's system, in mg/ml, t minutes after it is administered is given by the following formula:

$$T(t) = \begin{cases} at^3 & \text{for } 0 \leqslant t \leqslant 5 \\ b(t-6)^2 + 10 & \text{for } 5 < t \geqslant 7 \end{cases}$$

where a and b are constants.

(a) Give a formula for the tangent line to the graph of T at $t = 1$.

(b) Given that $T(t)$ is differentiable, find a and b. Give your answers in exact form.