Math 412. Adventure sheet on Compass and straightedge constructions

CONSTRUCTIONS WITH COMPASS AND STRAIGHTEDGE: Athena gives you two marked points in the plane; we call them $(0, 0)$ and $(1, 0)$. You are allowed to do three things:

- use a straightedge to draw the line between two marked points
- use the compass to draw a circle whose center is a marked point, and with a radius to another marked point
- mark any point of intersection between lines and circle you draw.

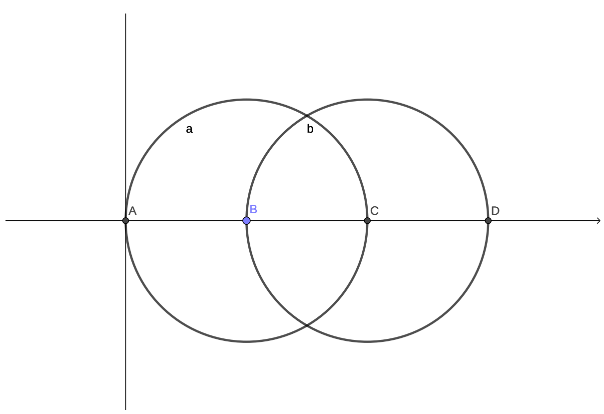
BASIC CONSTRUCTIONS:

- double or triple a length
- halve a length
- draw a perpendicular line through a point
- bisect an angle

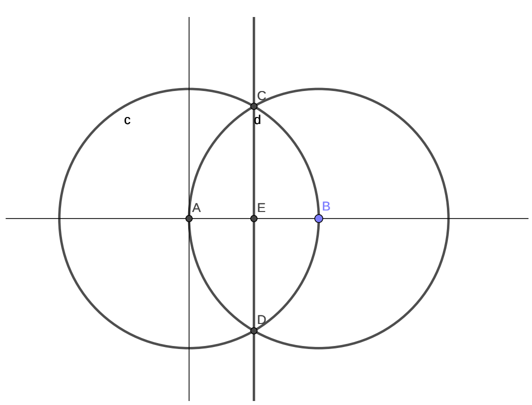
ADVANCED CONSTRUCTIONS:

- draw a parallel line though a point
- moving a segment of a given length onto a given line starting at a given point
- add or subtract two lengths
- create (α, β) from $(\alpha, 0)$ and $(\beta, 0)$
- create $(\alpha, 0)$ and $(\beta, 0)$ from (α, β)
- take the quotient of two lengths
- multiply two lengths
- take the square root of a length

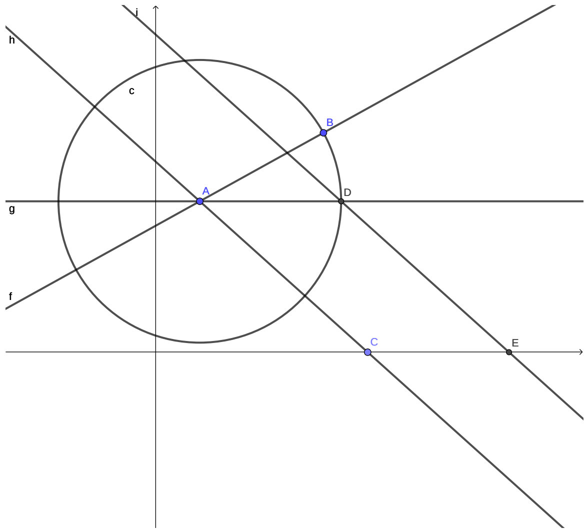
Here are examples of some of these constructions. Think about the rest or look them up in the book.



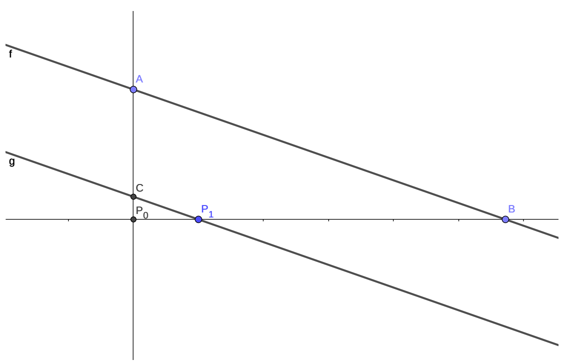
To double the length AB, make a circle centered at B passing through A, and intersect it with the line of AB. AC has twice the length.



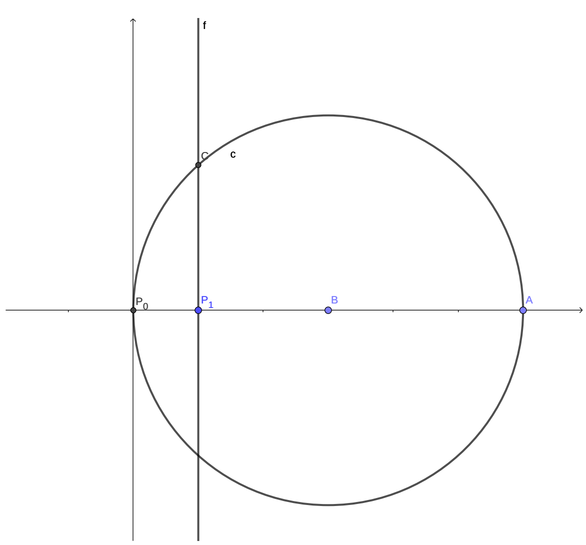
To halve the length AB, make a circle centered at A passing through B and a circle centered at B passing through A. These intersect at two points C and D. The line through CD meets the segment AB at its midpoint.



To move the segment AB to the x-axis starting at C, make a line g parallel to the x-axis passing through A. Make a circle centered at A passing through B, and mark the point of intersection with g; call it D. Finally, make a line parallel to AC passing through D. The segment CE has the same length as AB.



To make a segment whose length is the quotient of the lengths of two other segments, we can assume the given segments are on the y-axis (P_0A) and x-axis (P_0B). Remember that we have the point $P_1 = (1, 0)$ given. Make a line parallel to AB through P_1, and take its point of intersection with the y-axis; call it C. The length of P_0C is $\frac{|P_0A|}{|P_0B|}$.



To make a segment whose length is the square root of P_0A, first halve the segment; call the midpoint B. Take the circle e with center B passing through A. Make a line f parallel to the y-axis passing through $P_1 = (1, 0)$. Mark the intersection point C of e and f. The length of the segment P_1C is a the square root of P_0A.

DEFINITION: If we can mark a point $P = (\alpha, \beta)$ by using these rules repeatedly, we say that P is **constructible**. We say that a number α is **constructible** if $Q = (\alpha, 0)$ is constructible.

Our advanced constructions prove the following theorem (discuss!):

THEOREM 1:

(1) A point $P = (\alpha, y)$ is a constructible point if and only if α and β are constructible numbers.
(2) If x and y are constructible numbers, then so are $\alpha + \beta$, $\alpha - \beta$, $\alpha\beta$, α/β, and $\sqrt{\alpha}$ (if $\alpha > 0$).

DEFINITION: Let $\mathbb{F} \subseteq \mathbb{R}$ be a subfield. A **quadratic extension field** of $\mathbb{F}$ is a set of the form

$$\mathbb{F}(\sqrt{k}) = \{a + b\sqrt{k} \mid a, b \in \mathbb{F}\} \subseteq \mathbb{R}$$

for some $k \in \mathbb{F}$, $k > 0$, such that k is not a square of an element in $\mathbb{F}$.

DEFINITION: A **quadratic extension tower** over $\mathbb{Q}$ is a sequence of subfields of $\mathbb{R}$

$$\mathbb{Q} \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq F_t \subseteq \mathbb{R}$$

such that

$$F_1 = \mathbb{Q}(\sqrt{k_1}),\ F_2 = F_1(\sqrt{k_1}),\ \ldots\ ,\ F_t = F_{t-1}(\sqrt{k_t}),$$

with $k_1 \in \mathbb{Q}_{>0},\ \ k_2 \in (F_1)_{>0},\ \ \ldots\ \ ,\ \ k_t \in (F_{t-1})_{>0}.$

THEOREM 2: A number $\alpha \in \mathbb{R}$ is constructible if and only if there is a quadratic extension tower over $\mathbb{Q}$ for which $\alpha \in F_t$ (with notation as above).

THEOREM 3: If α is a root of an irreducible cubic polynomial in $\mathbb{Q}[x]$, then α is not an element of any field in a quadratic extension tower over $\mathbb{Q}$.

A. DOUBLING THE CUBE: Can you construct the base of a cube C with volume 2?

(1) Explain why $\sqrt[3]{2}$ is a root of an irreducible cubic polynomial in $\mathbb{Q}[x]$.
(2) Explain how it follows from Theorems 2 and 3 that it is impossible to double the cube with straightedge and compass.

B. TRISECTING AN ANGLE: Given an angle, can you divide it into three equal angles?

(1) To show this is impossible, why does it suffice to show that the number $\cos(20°)$ is not constructible?
(2) The triple-angle formula for cosine says that $\cos(3\theta) = 4\cos(\theta)^3 - 3\cos(\theta)$. Show that $\cos(20°)$ is a root of the polynomial $f(x) = 8x^3 - 6x - 1$.
(3) Show that $f(x)$ has no rational roots.[1] Conclude that $f(x)$ is irreducible in $\mathbb{Q}[x]$.
(4) Explain how it follows from Theorems 2 and 3 that it is impossible to trisect an angle with straightedge and compass.

[1] Hint: Suppose there is a rational root a/b in lowest terms. Plug in this root, clear denominators, and show that if a prime divides a, it divides b, so WLOG $a = 1$. Now show that if $p|b$ then $p|a$ or else $p = 2\ldots$

C. QUADRATIC EXTENSION FIELDS: Let $\mathbb{F} \subseteq \mathbb{R}$ be a subfield.

(1) Show that[2] any quadratic extension field $\mathbb{F}(\sqrt{k})$ is a subfield of $\mathbb{R}$.
(2) Show that if $x \in \mathbb{R}$ is a solution of $Ax^2 + Bx + C = 0$ for some $A, B, C \in \mathbb{F}$, then $x \in \mathbb{F}(\sqrt{k})$ for some k.
(3) Show that the map $\phi : \mathbb{F}(\sqrt{k}) \to \mathbb{F}(\sqrt{k})$ given by $\phi(a + b\sqrt{k}) = a - b\sqrt{k}$ is a ring homomorphism, and that $\phi(f) = f$ for any element of $\mathbb{F}$.
(4) Use this fact to show that if $f(x)$ is a cubic polynomial with coefficients in $\mathbb{F}$, and $f(a + b\sqrt{k}) = 0$, then $f(a - b\sqrt{k}) = 0$.[3]

D. INTERSECTION POINTS: Let $\mathbb{F} \subseteq \mathbb{R}$ be a subfield. Let L_1 and L_2 be lines through two points with coordinates in $\mathbb{F}$. Let C_1 and C_2 be circles whose centers have coordinates in $\mathbb{F}$, and radii are values of $\mathbb{F}$.

(1) If L_1 is not vertical, why are the slope and y-intercept of L_1 values of $\mathbb{F}$? What can you say about the equation of L_1 if it is vertical?
(2) Explain why C_1 has an equation of the form $(x - A)^2 + (y - B)^2 = C^2$, where $A, B, C \in \mathbb{F}$.
(3) Explain why the intersection point of L_1 and L_2 (if they are not parallel) has coordinates in $\mathbb{F}$.
(4) Explain why the intersection points of L_1 and C_1 (if they exist) have coordinates in a quadratic extension field of $\mathbb{F}$.
(5) Explain why the intersection points of C_1 and C_2 (if they exist) have coordinates in a quadratic extension field of $\mathbb{F}$.

E. CONSTRUCTIBLE NUMBERS:

(1) Explain why every rational number $r \in \mathbb{Q}$ is constructible.
(2) Explain why any number in a quadratic extension field of $\mathbb{Q}$ is constructible.
(3) Show that, if every number in a subfield $\mathbb{F}$ of $\mathbb{R}$ is constructible, then every number in any quadratic extension field $\mathbb{F}(\sqrt{k})$ of $\mathbb{F}$ is constructible.
(4) Show that any element of $r \in F_t$ for a field in a quadratic extension tower over $\mathbb{Q}$ is constructible.
(5) Show that any constructible number $r \in \mathbb{R}$ is an element of some field F_t that lies in a quadratic extension tower.
(6) Conclude the proof of Theorem 2.

F. CUBIC POLYNOMIALS AND QUADRATIC EXTENSIONS:

(1) Show that if $\mathbb{F}$ is a field, $\mathbb{F}(\sqrt{k})$ is a quadratic extension, and $\alpha \in \mathbb{F}(\sqrt{k})$, then $g(x) = (x - \alpha)(x - \phi(\alpha))$ has coefficients in $\mathbb{F}$ (i.e., is a polynomial in $\mathbb{F}[x]$), where ϕ is the map from the Quadratic extension fields problem.
(2) Show that if $\mathbb{F}$ is a field, $\mathbb{F}(\sqrt{k})$ is a quadratic extension, $f(x) \in \mathbb{F}[x]$ is a cubic polynomial, and $f(x)$ has a root in $\mathbb{F}(\sqrt{k})$, then $f(x)$ has a root in $\mathbb{F}$.[4]
(3) Show that if γ is a root of an irreducible cubic polynomial in $\mathbb{Q}[x]$, then γ is not an element of any field in a quadratic extension tower over $\mathbb{Q}$.[5]
(4) Conclude the proof of Theorem 3.

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[2]Hint: What is $(a + b\sqrt{k})(\frac{a - b\sqrt{k}}{a^2 - b^2 k})$?

[3]Hint: Let $\alpha = a + b\sqrt{k}$, and compute $\phi(f(a + b\sqrt{k}))$.

[4]Hint: Show that the polynomial $g(x)$ from the previous part divides $f(x)$.

[5]Hint: To obtain a contradiction, suppose γ is constructible, take a quadratic extension tower and pick F_t such that F_t contains γ but F_{t-1} does not.