Math 412 Adventure sheet on cosets

DEFINITION: Fix a group G and a subgroup K. A **right K-coset** of K is any subset of G of the form

$$K \circ b = \{k \circ b \mid k \in K\}$$

where $b \in G$. Similarly, a **left K-coset** of K is any set of the form $b \circ K = \{b \circ k \mid k \in K\}$.

PROPOSITION: Fix a group G and a subgroup K. The total number of right K-cosets is equal to the total number of left K-cosets.

DEFINITION: Fix a group G and a subgroup K. The **index** of K in G is the total number of *distinct* right K-cosets of K in G. We write this index $[G : K]$.

LAGRANGE'S THEOREM: Fix a group G and a subgroup K. Then $|G| = |K|[G : K]$.

DEFINITION: Let $a, b \in G$. We say a is **congruent** to b modulo K if $ab^{-1} \in K$.

A. EXAMPLE IN THE GROUP OF INTEGERS. Let $G = (\mathbb{Z}, +)$ and let K be the subgroup generated by 7.

(1) Verify that $K = 7\mathbb{Z} = \{7k | k \in \mathbb{Z}\}$.
(2) Describe the right K-coset $K + 0$.
(3) Explain why the left/right K-coset containing a is the same as the set $[a]_7 \subseteq \mathbb{Z}$.
(4) Find the index $[G : K]$. Verify LaGrange's theorem.

B. EXAMPLE IN S_3. Consider the subgroup K of S_3 generated by $(1\,2)$.

(1) List out all the elements of K. What does Lagrange's Theorem predict about the number of right cosets of K?
(2) Find the right K-coset Ke. Show that it is the same as the right coset $K(1\,2)$.
(3) Find the right coset $K(2\,3)$. Show that it is the same as the right coset $K(1\,2\,3)$.
(4) Find the right coset $K(1\,3)$. Show that it is the same as the right coset $K(1\,3\,2)$.
(5) Write out all the elements of S_3 explicitly, grouping them together if they are in the same right K-coset.
(6) Express S_3 as a disjoint union of right K-cosets. How many right K-cosets are there in total?
(7) Verify Lagrange's Theorem for $K \subseteq S_3$.

C. RIGHT K-COSETS AND CONGRUENCE MODULO K. Fix a group G and a subgroup K.

(1) Prove that a is congruent to b modulo K if and only if $a \in Kb$. So the set of all elements congruent to b mod K is precisely the right coset Kb.
(2) Prove that congruence modulo K is an equivalence relation.
(3) Discuss: the concept of right K-coset is the group analog of the concept of congruence class modulo an ideal for rings.
(4) Show that if $b \in Ka$, then $Ka = Kb$. Show also that if $b \notin Ka$, then $Ka \cap Kb = \emptyset$. That is, two cosets are either exactly the same subset of G or they do not overlap at all.

D. THE PROOF OF LAGRANGE'S THEOREM. Fix a group G and a subgroup K. Let $a, b \in G$.

(1) Prove that there is a bijection

$$Ka \to Kb$$

given by right multiplication by $a^{-1}b$.
(2) Prove that G is the disjoint union of its distinct right K-cosets, all of which have cardinality $|K|$.
(3) Prove that if G is finite, then $|G| = [G : K]|K|$.

(4) Conclude that the order of any subgroup K must divide the order of G.
(5) Conclude that the order of any element in G must divide the order of G.

E. LEFT VS RIGHT COSETS. Let G be a group and K be a subgroup of G.

(1) With the notation we used in A, is $K + 0 = 0 + K$? How about $K + a$ and $a + K$ for some $a \in \mathbb{Z}$?
(2) With the notation we used in B, is $K(1\,2\,3) = (1\,2\,3)K$?
(3) TRUE OR FALSE: In an arbitrary group G, for any subgroup K, $Kg = gK$ for all $g \in K$.
(4) TRUE OR FALSE: In an arbitrary abelian group G, for any subgroup K, $Kg = gK$ for all $g \in K$.
(5) TRUE OR FALSE: In an arbitrary group G, every right K-coset is a subgroup of G.

F. Fix a subgroup K of a group $(G, \circ)$.

(1) Show that $Ke = K = eK$.
(2) Show that for any $a \in G$, there is a bijection $K \longrightarrow Ka$.
(3) Prove that $|K \circ a| = |a \circ K|$, even if in general $K \circ a \neq \circ K$.
(4) Prove that if G is finite, the number of left K-cosets is the same as the number of right K-cosets.

G. A CAUTIONARY EXAMPLE. Let G be a group and let K be a subgroup. Consider the set G/K of all right K-cosets. It is tempting to try to define a quotient group as we defined quotient rings. That is, we can try to define a binary operation $\star$ on G/K by $(K \circ g) \star (K \circ h) := K(g \circ h)$.

(1) Show that in the example of $7\mathbb{Z}$ in $\mathbb{Z}$ from A, $\star$ is a well-defined binary operation.
(2) Show that in the example of $K = \langle(1\,2)\rangle$ in S_3 as in B, $\star$ is **not** a well-defined binary operation. In fact, there is *no natural way to induce a quotient group structure on the set of cosets* G/K.
(3) For R_4 in D_4 in A, is $\star$ a well-defined binary operation on the set of right cosets D_4/R_4? Is $(D_4/R_4, \star)$ a group?

H. A MATRIX EXAMPLE. Consider $G = GL_2(\mathbb{R})$, the subgroup $K = SL_2(\mathbb{R})$, and $A = \begin{bmatrix} 1 & 17 \\ 0 & \pi \end{bmatrix}$.

(1) Prove that the right K-coset KA in $GL_2(\mathbb{R})$ is $\{B \in GL_2(\mathbb{R}) \mid \det B = \pi\}$.
(2) Prove that the left K-coset $AK = KA$.
(3) Prove that the right K-cosets KC and KD are the same in this case if and only if $\det C = \det D$.
(4) What is the index $[GL_2(\mathbb{R}) : SL_2(\mathbb{R})]$?