Math 412
Final Exam review problems

True or false. Justify!

1) $D_3 \cong \mathbb{Z}_2 \times \mathbb{Z}_3$. FALSE
2) $D_3 \cong \mathcal{S}_3$. TRUE
3) $D_4 \cong \mathcal{S}_4$. FALSE
4) $\mathbb{Z}_2 \times \mathbb{Z}_2 \cong \mathbb{Z}_4$. FALSE
5) $\mathbb{Z}_3 \times \mathbb{Z}_2 \cong \mathbb{Z}_6$. TRUE
6) An element g of a group G can satisfy $g^{30} = e$ and have order 6. TRUE
7) An element g of a group G can satisfy $g^{30} = e$ and have order 7. FALSE
8) Every normal subgroup is the kernel of some group homomorphism. TRUE
9) If a group G contains an element of infinite order, then G is infinite. TRUE
10) If a group G contains a nontrivial element of finite order, then G is finite. FALSE
11) If every element in a group G has finite order, then G is finite. FALSE
12) There are no nontrivial simple abelian groups. FALSE
13) There exists a group G of order 8 acting on a set X such that some $x \in X$ has orbit of cardinality 5. FALSE
14) There exists a group G of order 12 acting on a set X such that some $x \in X$ has orbit of cardinality 6 and stabilizer of order 4. FALSE
15) When a group G acts on a set X, a point $x \in X$ is fixed if and only if $\mathrm{Stab}(x) = G$. TRUE
16) For every $n \geqslant 5$, $\mathcal{A}_n$ is simple. TRUE
17) The group of units in $\mathbb{Z}$ is cyclic. TRUE
18) The group of units in $\mathbb{Z}_{51}$ is cyclic. FALSE
19) The group of units in $\mathbb{Z}_8$ is cyclic. FALSE
20) The subgroup of $SL_2(\mathbb{R})$ generated by $\begin{pmatrix} \cos(2\pi/n) & -\sin(2\pi/n) \\ \sin(2\pi/n) & \cos(2\pi/n) \end{pmatrix}$ is isomorphic to $\mathbb{Z}_n$. TRUE
21) There always exists a group homomorphism between any two groups. TRUE
22) If H is an abelian subgroup of the (possibly nonabelian) group G, then H is normal. FALSE
23) If H is a subgroup of an abelian group G, then G/H is abelian. TRUE
24) If a group G of order 14 acts on a set with 14 elements, it's possible the total number of orbits is 3. FALSE
25) When the Klein-4 group acts on a set of 11 elements, there are at most 4 orbits. FALSE
26) The center of an abelian group G is the set of all elements of G. TRUE
27) The center of a group G is an abelian group. TRUE
28) The center of a group is always a normal subgroup. TRUE
29) If every proper subgroup of a group G is cyclic, then G is cyclic. FALSE
30) There exists a surjective group homomorphism $\mathbb{Z}_7 \longrightarrow \mathbb{Z}_5$. FALSE
31) There exists an injective group homomorphism $\mathcal{S}_7 \longrightarrow \mathcal{S}_8$. TRUE
32) There exists an injective group homomorphism $\mathbb{Z}_7 \longrightarrow \mathbb{Z}_8$. FALSE
33) When a nontrivial group acts on itself by conjugation, there is always a fixed point. TRUE
34) When a nontrivial group acts on itself by left multiplication, there is always a fixed point. FALSE
35) If an action of the group G on the set X has at least one fixed point, then the action is faithful. FALSE
36) If an action of the group G on the set X has no fixed points, the action is faithful. FALSE
37) The quotient group G/K is a subset of G. FALSE
38) The elements gK and hK are equal in G/K if and only if $g = h$. FALSE

39) If G is a group of order n and $k|n$, there is an element of G of order k. FALSE
40) If G is a group of order n and $k|n$, there is a subgroup of G of order k. FALSE
41) Every element in $\mathcal{S}_{123}$ is a product of elements of order 2. TRUE
42) Every element in $\mathcal{S}_{123}$ is a product of elements of order 3. FALSE
43) There exists a field $\mathbb{F}$ such that both $(\mathbb{F}, +)$ and $(\mathbb{F}^\times, \times)$ are cyclic. TRUE
44) Every quotient of a nonabelian group is nonabelian. FALSE
45) A subgroup of order 2 is always normal. FALSE
46) Every subgroup of index 2 is normal. TRUE
47) Every subgroup of index 3 is normal. FALSE
48) For a ring homomorphism $f\colon R \to S$, $f^\times\colon R^\times \to S^\times$ given by $f^\times(u) = f(u)$ is a group homomorphism. TRUE
49) The image if a group homomorphism $G \longrightarrow H$ with G abelian is always an abelian subgroup of H. TRUE
50) If there exists a nontrivial group homomorphism $G \longrightarrow H$ with G abelian, then H is abelian. FALSE
51) Suppose that G acts on itself by conjugation. Then it's not necessary that every point be a fixed point. TRUE
52) The quotient group $\mathbb{Q}/\mathbb{Z}$ is a finite group. FALSE
53) Let $\mathbb{F}$ be a finite field with no nonidentity element g satisfying $g^2 = 1$. Then $|\mathbb{F}^\times|$ is odd. FALSE
54) There exists a surjective group homomorphism $\mathcal{S}_5 \longrightarrow \mathbb{Z}_3$. FALSE
55) If $g^{18} = e$, then the order of g is 18. FALSE
56) A group of order 400 can have an element of order 19. FALSE
57) The intersection of two normal subgroups is a normal subgroup. TRUE
58) If R is a finite ring, $R^\times$ is a cyclic group. FALSE
59) Given any ring $(R, +, \times)$, $(R, +)$ is always a group. TRUE
60) Given any ring $(R, +, \times)$, $(R, \times)$ is always a group. FALSE
61) The rings $\mathbb{Q}[x]/(x^2)$ and $\mathbb{Q}[x]/(x^2 + 1)$ are isomorphic. FALSE
62) $\mathcal{S}_3$ is a cyclic group. FALSE
63) There are two nonisomorphic cyclic groups of order 20. FALSE
64) There exists a subgroup of S_5 that is isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3$. FALSE
65) Every 4-cycle in $\mathcal{S}_{103}$ is odd. TRUE
66) $\mathcal{S}_{120}$ has no subgroup isomorphic to D_{60}. FALSE
67) Every group of order 12 contains an element of order 4. FALSE
68) Every group of order 120 contains an element of order 3. TRUE
69) Let R be the subgroup of all rotations in D_4. Then $D_4/R \cong \mathbb{Z}_3^\times$. TRUE
70) Given a group G and $x \in G$, x defines a group homomorphism $G \longrightarrow G$ by $g \mapsto gxg^{-1}$. FALSE
71) If F is a field and R is a nonzero ring, every ring homomorphism $F \longrightarrow R$ is injective. TRUE
72) If G and H are two groups of the same order, then $G \cong H$. FALSE
73) Every group of order 29 is simple. TRUE
74) The image of a group homomorphism is always a normal subgroup. FALSE
75) The kernel of a group homomorphism is always a normal subgroup. TRUE
76) Every finite group G is isomorphic to a subgroup of $\mathcal{S}_n$ for some n. TRUE
77) Every quotient of a domain is a domain. FALSE
78) Every quotient of a field is a field. TRUE
79) In any group G, the product of elements of finite order always has finite order. FALSE
80) Every nontrivial group has at least two subgroups. TRUE
81) Every nontrivial group has at least two normal subgroups. TRUE
82) Every ring homomorphism $M_2(\mathbb{R}) \to R$ to a nontrivial ring R is injective. TRUE
83) There are exactly 2 ring homomorphisms $\mathbb{Z}_2 \times \mathbb{Z}_2$ to $\mathbb{Z}_4$. FALSE
84) Every subgroup of an abelian group is abelian. TRUE

85) There are no group homomorphisms $\mathbb{Z}_2 \to \mathbb{Z}_4$. FALSE
86) There are no group homomorphisms $\mathbb{Z}_n \to \mathbb{Z}$. FALSE
87) In $\mathbb{Z}$, if $n = p_1 \cdots p_t = q_1 \cdots q_s$, for primes p_i, q_j, then $s = t$ and $p_1 = q_1, \ldots, p_s = q_s$. FALSE
88) In general, the fastest way to find the gcd of two large integers is to factor them into primes. FALSE
89) The equation $[a]_n x = [b]_n$ has a solution in $\mathbb{Z}_n$ if and only if $\gcd(a, n) = 1$. FALSE
90) The system of equations $7|(x+3)$ and $11|(x-1)$ has a solution modulo 77. TRUE
91) The system of equations $3|x$ and $6|(x-1)$ has a solution modulo 18. FALSE
92) If $n|a$ and $m|a$, then $nm|a$. FALSE
93) Given any ring R, there exists exactly one ring homomorphism $\mathbb{Z} \longrightarrow R$. TRUE
94) Given any ring R, there exists exactly one ring homomorphism $R \longrightarrow \mathbb{Z}$. FALSE
95) Given any ring R, there exists exactly one ring homomorphism $\mathbb{Z}_n \longrightarrow R$. FALSE
96) Given any ring R, there exists exactly one ring homomorphism $R \longrightarrow \mathbb{Z}_n$. FALSE
97) Every element in $\mathbb{Z}$ is a unit. FALSE
98) The additive inverse of $[5]_{77}$ in $\mathbb{Z}_{77}$ is $[149]_{77}$. TRUE
99) The multiplicative inverse of $[5]_{77}$ in $\mathbb{Z}_{77}$ is $[108]_{77}$. TRUE
100) Every nonzero ring contains at least two ideals. TRUE
101) Every domain is a field. FALSE
102) Every field is a domain. TRUE
103) The zero ring is a domain. FALSE
104) There always exists a ring homomorphism between any two rings. FALSE
105) Any commutative ring that has only two ideals is a field. TRUE
106) The kernel of any ring homomorphism is an ideal. TRUE
107) The kernel of any ring homomorphism is a subring. FALSE
108) The image of any ring homomorphism is an ideal. FALSE
109) The image of any ring homomorphism is a subring. TRUE
110) If R is a commutative ring and $(g) = R$, then g is a unit. TRUE
111) If R is a domain, then $R[x]$ is a domain. TRUE
112) If F is a field, then $F[x]$ is a field. FALSE
113) Every reducible polynomial of degree 4 in $F[x]$ for a field F has a root in F. FALSE
114) Every reducible polynomial of degree 3 in $F[x]$ for a field F has a root in F. TRUE
115) If $p(x) \in \mathbb{Z}_2[x]$ has degree 3, then $\mathbb{Z}_2[x]/(p(x))$ has 4 elements. FALSE
116) If a monic $p(x) \in F[x]$ for some field F is irreducible, $\gcd(p(x), f(x))$ is 1 or p for any f. TRUE
117) If F is a field, the remainder of dividing $f(x)$ by $x - a$ is $f(a)$. TRUE
118) Modern algebra is fun! TRUE
119) The ring $\mathbb{Z}_n[x]$ is a domain. FALSE
120) $\mathbb{Z}_{12} \times \mathbb{Z}_5 \cong \mathbb{Z}_{60}$ as rings. TRUE
121) $\mathbb{Z}_{10} \times \mathbb{Z}_6 \cong \mathbb{Z}_{60}$ as rings. FALSE
122) If f and g differ by a unit in $F[x]$, where F is a field, then $(f, g) = 1$. TRUE
123) If $uf + vg = 4$ in $\mathbb{Q}[x]$, then $f + (g)$ is a unit in $\mathbb{Q}[x]/(g)$. TRUE
124) In $R[x]$, the product of two monic polynomials can be zero. FALSE
125) For a field F, $F[x] \to F$ sending each polynomial to its constant term is a ring homomorphism. TRUE
126) $x^3 + 2$ is a unit in $\mathbb{Z}_5[x]/(x^4 - x^2)$. TRUE
127) The quotient ring $\mathbb{R}[x]/(x^3 - x - 6)$ is a field. FALSE
128) Every ideal is the kernel of some ring homomorphism. TRUE
129) Any subring of a domain is a domain. TRUE
130) Any subring of a field is a field. FALSE
131) $2^3 \equiv 2^7 \mod 5$. TRUE
132) Every integer is congruent to the sum of its digits modulo 11. FALSE
133) An element of a commutative ring $R \neq \{0\}$ cannot be both a unit and a zerodivisor. TRUE

134) A subset of a ring that is also a ring is a subring. FALSE
135) $\mathbb{Z}_n$ is a domain if and only if it is a field. TRUE
136) If $ua + vb = n$ for some $a, b, u, v \in \mathbb{Z}$, then $(a, b) = n$. FALSE
137) If $ua + vb = 1$ for some $a, b, u, v \in \mathbb{Z}$, then $(a, b) = 1$. TRUE
138) Every element in $\mathbb{Z}_{11}$ is invertible. FALSE
139) In $\mathbb{Z}_{77}$, $(a) = (b)$ if and only if $a = b$. FALSE
140) Every ideal in $\mathbb{Z}_{123}$ is principal. TRUE
141) In $\mathbb{Z}[x]$, $(a, b) = (\gcd(a, b))$. FALSE
142) If R and S are domains, then $R \times S$ is a domain. FALSE
143) In any ring R, $ab = 0$ implies $a = 0$ or $b = 0$. FALSE
144) In any ring R, we can cancel addition. TRUE
145) In any ring R, we can cancel multiplication. FALSE
146) On the set of real numbers, $r \sim s$ if and only if $|r| = |s|$ defines an equivalence relation. TRUE
147) If a is even and b is odd, (a, b) is even. FALSE
148) If $a|b$ and $b|c$, then $a|c$. TRUE
149) If I and J are ideals in a ring R, $I \cup J$ is an ideal in R. FALSE
150) There is a bijection between ideals in R containing I and ideals in R/I. TRUE
151) Every prime ideal is maximal. FALSE
152) If I is a prime ideal, then R/I is a field. FALSE
153) If R is a field, then R has at most two ideals. TRUE
154) If p is a prime and a is any integer, $a^{p-1} \equiv 1 \mod p$. FALSE
155) If p is a prime and a is any integer, $a^p \equiv a \mod p$. TRUE