Math 412. Quotient Groups and the First Isomorphism Theorem

FIRST ISOMORPHISM THEOREM FOR GROUPS: Let $G \xrightarrow{\phi} H$ be a *surjective* group homomorphism with kernel K. Then $G/K \cong H$. More precisely, the map

$$G/K \xrightarrow{\overline{\phi}} H \qquad gK \mapsto \phi(g)$$

is a well-defined group isomorphism.

A group G is called **simple** if the only normal subgroups of G are $\{e\}$ and G itself.

A. Use the first isomorphism theorem to prove the following:

(1) For any field $\mathbb{F}$, the group $SL_n(\mathbb{F})$ is normal in $GL_n(\mathbb{F})$ and the quotient $GL_n(\mathbb{F})/SL_n(\mathbb{F})$ is isomorphic to $\mathbb{F}^\times$.

(2) For any n, the group A_n is normal in $\mathcal{S}_n$ and the quotient $\mathcal{S}_n/A_n$ is cyclic of order two.

B. Let H be the subgroup of $\mathbb{Z} \times \mathbb{Z}$ generated by $(5, 5)$.

(1) Find an element of finite order and an element of infinite order in the quotient $\mathbb{Z} \times \mathbb{Z}/H$.

(2) Prove that

$$\psi : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}_5 \quad (m, n) \mapsto (m - n, [n]_5)$$

is a group homomorphism.

(3) Prove that ψ is surjective and compute its kernel.

(4) Show that $(\mathbb{Z} \times \mathbb{Z})/H$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}_5$.

C. Consider the map $(\mathbb{C}^\times, \times) \to (\mathbb{R}_{>0}, \times)$ sending z to $|z|$. Show that this is a surjective group homomorphism with kernel the circle group S^1. Describe the quotient $\mathbb{C}^\times/S^1$.

D. THE FIRST ISOMORPHISM THEOREM.

(1) Prove the First Isomorphism Theorem for Groups.

(2) What does the theorem say about group homomorphisms that are not necessarily surjective?

(3) Prove that if $f : G \longrightarrow H$ is a group homomorphism, then $|f(G)| \,\big|\, |G|$.

E. SIMPLE GROUP WARMUP. Consider the groups $\mathbb{Z}$, $\mathbb{Z}_{35}$, $GL_5(\mathbb{Q})$, S_{17}, D_{100}. Find nontrivial normal subgroups for each of them. Are these groups simple?

F. EASY PROOFS. Let G be an arbitrary group.

(1) Prove that if G is simple, then every nontrivial[1] homomorphism $G \to H$ is *injective*.

(2) Prove that if G is simple, then every surjective homomorphism $G \to H$ is an isomorphism.

(3) If $G = H \times K$, where neither H nor K is trivial, then G is *not* simple.

[1]By nontrivial, we mean the map does not send every element to e.

If a finite group G is not simple, there exists some nontrivial proper normal subgroup K of G and K and G/K are both smaller than G. We can think of G as being "made from" the smaller groups K and G/K. From this point of view, the simple groups are the basic building blocks, since we can't write them as "made from" smaller groups in this way.

THEOREM: If G is a finite abelian group, then G is simple if and only if it is cyclic of prime order.

THEOREM: The alternating groups $\mathcal{A}_n$ for $n \geqslant 5$ are simple.

There is a classification of all finite simple groups. It consists of a few infinite families (like $\{\mathbb{Z}_p \mid p \text{ prime}\}$ and $\{\mathcal{A}_n \mid n \geqslant 5\}$ and a few additional *sporadic* simple groups which do not belong to any of the families. The UM Math department played a big role in this classification: one of the sporadic simple groups is named after former department chair Jack McLaughlin, and the final and largest sporadic simple group, the Monster group, was discovered by current faculty member Robert Griess.

G. Prove the first theorem above: If G is a finite abelian group, then G is simple if and only if it is cyclic of prime order.[2]

H. PRODUCTS AND QUOTIENT GROUPS: Let K and H be arbitrary groups and let $G = K \times H$.

(1) Find a natural homomorphism $G \to H$ whose kernel K' is $K \times e_H$. Prove that $K \cong K'$.
(2) Prove that K' is a normal subgroup of G, whose cosets are all of the form $K \times h$ for $h \in H$.
(3) Prove that G/K' is isomorphic to H.

I. APPLICATIONS OF SIMPLICITY OF $\mathcal{A}_n$, $n \geqslant 5$:

(1) Let $n \geqslant 5$ and $m \leqslant n$ be odd. Show that $\mathcal{A}_n$ is generated by m-cycles.[3]
(2) Show that if $n \geqslant 5$ and $m < n$, then there is no nontrivial action of $\mathcal{A}_n$ on a set of m elements.
(3) Show that if $n \geqslant 5$ and $2 < m < n$, then there is no action of $\mathcal{S}_n$ on a set of m elements that has only one orbit.
(4) Show that the previous statement is false if $n = 4$.

[2] Hint: Reuse something you showed on the homework!

[3] Hint: Show that the subgroup of $\mathcal{A}_n$ generated by m-cycles is normal.