Math 412. Adventure sheet on the First Isomorphism Theorem

NOETHER'S FIRST ISOMORPHISM THEOREM: *Let $R \xrightarrow{\phi} S$ be a surjective homomorphism of rings. Let I be the kernel of ϕ. Then R/I is isomorphic to S.*

A: Fix any real number a. Consider the evaluation map

$$\eta : \mathbb{R}[x] \to \mathbb{R} \qquad f \mapsto f(a)$$

(1) Understand why the evaluation map is a **surjective ring homomorphism.**
(2) Prove[1] that the kernel of η is the ideal $I = (x-a)$ of $\mathbb{R}[x]$ generated by $x-a$.
(3) Use the first isomorphism theorem to prove that $\mathbb{R}[x]/(x-a)$ is isomorphic to $\mathbb{R}$.
(4) Give a direct proof that $\mathbb{R}[x]/(x-a) \cong \mathbb{R}$ by thinking about the congruence classes $f + (x-a)$.[2] Why is there a bijection with $\mathbb{R}$ that preserves the ring structure?

B: Let i be the complex number $\sqrt{-1}$. Consider the ring homomorphism

$$\phi : \mathbb{R}[x] \to \mathbb{C} \qquad f \mapsto f(i)$$

(1) Prove that ϕ is **surjective**.
(2) Prove that $x^2+1 \in \ker\phi$.
(3) Prove that the kernel contains no (nonzero) polynomial of degree less than two.
(4) Prove that x^2+1 generates $\ker\phi$. [Hint: If $f(x)$ is in the kernel, use the division algorithm to divide f by x^2+1 and see what happens under ϕ.]
(5) Use the First Isomorphism Theorem to explain how to think about the complex numbers as a quotient of the polynomial ring $\mathbb{R}[x]$.

C: PROOF OF THE FIRST ISOMORPHISM THEOREM. Fix a surjective ring homomorphism $\phi : R \to S$. Let I be its kernel, Define $\overline{\phi} : R/I \to S$ by $\overline{\phi}(r+I) = \phi(r)$.

(1) Show that $\overline{\phi}$ is a well-defined map.
(2) Show that $\overline{\phi}$ is a surjective ring homomorphism.
(3) Show that $\overline{\phi}$ is injective.
(4) Prove the First Isomorphism Theorem.

D: NEW PROOFS FOR OLD FACTS.

(1) Show that whenever n and m are relatively prime integers, $\mathbb{Z}_{mn} \cong \mathbb{Z}_n \times \mathbb{Z}_m$.[3]
(2) Let k be a field, $d \geqslant 1$, and $R = k[x_1, \dots, x_d]$. Show that $R/(x_1, \dots, x_d) \cong k$.

E: PRIME IDEALS. An ideal $P \subsetneq R$ in a commutative ring R is prime if $fg \in P$ implies $f \in P$ or $g \in P$.

(1) Show that an ideal P is a prime ideal if and only if R/P is a domain.
(2) What are the prime ideals in $\mathbb{Z}$?
(3) Show that the ideal (x, y) in $\mathbb{Z}[x, y]$ is prime.

[1]Hint for the harder direction: say $g \in \ker\eta$, and use the division algorithm to divide g by $x-a$; apply η.
[2]Hint: For quotient rings of polynomial rings over a field, every congruence class contains a unique [what?]
[3]We have done this in a problem set! But now we can give a new proof using the First Isomorphism Theorem.