Math 412 Adventure sheet on group actions

DEFINITION: Let $(G, \star)$ be a group. Let X be a set. A **group action** of G on X is a function

$$G \times X \to X \qquad\qquad (g, x) \to g \cdot x,$$

satisfying the axioms:

(1) $e_G \cdot x = x$ for all $x \in X$, and
(2) $h \cdot (g \cdot x) = (h \star g) \cdot x$ for all $g, h \in G$ and all $x \in X$.

DEFINITION: A group action of the group G on the set X is **faithful** if the only element $g \in G$ such that $g \cdot x = x$ for all $x \in X$ is the identity.

DEFINITION: Fix a group action of the group G on the set X. The **orbit** of an element $x \in X$ is the subset of X

$$O(x) := \{g \cdot x \mid g \in G\} \subseteq X.$$

A. Let D_4 be the symmetry group of the square. The group D_4 acts on the set of points X of the square in a canonical way. Note that X is an infinite set of points.

(1) Draw a picture of the square in the Cartesian plane so its vertices are $(\pm 1, \pm 1)$. Explain the canonical action of D_4 on the square.
(2) Compute the orbit of each the following types of points (and sketch): the origin, a vertex, a nonzero point on a diagonal of the square, a nonzero point on the horizontal axis of symmetry, a nonzero point not on any axis of symmetry.
(3) What is the largest number of points any orbit can have? Find an explicit point whose orbit achieves this value.
(4) True or False: under the given action of D_4 on the square, all vertices have the same orbit.
(5) True or False: the given action of D_4 on the square is faithful.

B. Let $\mathcal{S}_4$ be the group of permutations of the set $\{1, 2, 3, 4\}$. There is a canonical action of $\mathcal{S}_4$ on the set $X = \{1, 2, 3, 4\}$ defined by $g \cdot x = g(x)$.

(1) Verify that this is a group action.
(2) Find the orbit of the element $4 \in X$. Are any two elements of X in the same orbit?
(3) Let Y be the set of *subsets* of $\{1, 2, 3, 4\}$. Describe a natural action of $\mathcal{S}_4$ taking every subset of $\{1, 2, 3, 4\}$ to another. Quickly convince yourself that your action satisfies the axioms of an action.
(4) Find the orbit of the set $\{1\} \in Y$ under the action of S_4 you described in (3). What is the cardinality of this orbit?
(5) Find the orbit of $\{1, 2\} \in Y$. What is the cardinality of this orbit?
(6) A **fixed point** is a point $x \in Y$ such that $g \cdot x = x$ for all $g \in G$. Does Y have any fixed points under this action?
(7) The action of S_4 on Y partitions Y up into disjoint orbits. Describe these.

C. Fix a group action of a group G on a set X.

(1) Observe that if you fix an element $g \in G$, then the rule $x \to g \cdot x$ is a function from $X \to X$. We denote this function as $\mathrm{ad}(g) : X \to X$.
(2) Verify that $\mathrm{ad}(e)$ is the identity function on X.
(3) Show that, for $g, h \in G$, $\mathrm{ad}(g) \circ \mathrm{ad}(h) = \mathrm{ad}(gh)$.

(4) Show that ad is a group homomorphism from G to $\mathrm{Bij}(X)$, the group of bijections of X (under composition).
(5) Show that ad is injective if and only if the action is faithful.

D. CAYLEY'S THEOREM. Let G be a finite group of order n.
(1) Show that the rule $g \cdot x = gx$ defines a group action of G on itself ($X = G$).
(2) Show that this action is faithful.
(3) Conclude that G is isomorphic to a subgroup of $\mathcal{S}_n$.

E. Consider the group Cube of symmetries of the cube. Recall that $|\mathrm{Cube}| = 24$.
(1) Observe that Cube acts on the set of diagonals (from one vertex to its opposite) of the cube.
(2) Show that this action is faithful.[1]
(3) Show that Cube is isomorphic to $\mathcal{S}_4$.
(4) Conclude that the orders of the elements in Cube are exactly $1, 2, 3, 4$, and that Cube is generated by two elements.

F. There can be different actions of the same group G on the same set X. For example, the group $\mathbb{Z}_2$ can act on the Cartesian plane $\mathbb{R}^2$ as follows:

$$[0]_2 \cdot (x, y) = (x, y) \qquad [1]_2 \cdot (x, y) = (y, x).$$

A *different* group action is as follows:

$$[0]_2 \cdot (x, y) = (x, y) \qquad [1]_2 \cdot (x, y) = (-x, -y).$$

(1) Verify that these are both group actions. Describe them geometrically.
(2) Find another group action, different from these, of $\mathbb{Z}_2$ on $\mathbb{R}^2$. There are many possibilities; check on a neighboring group to see what they came up with as well.
(3) For each of the three actions in play here, describe the orbits. How many elements can be in an orbit?

G. Let a group G act on a set X.
(1) If G has n elements, explain why every orbit has at most n elements.
(2) If X has m elements, explain why every orbit has at most m elements.
(3) Prove that the relation "$x \sim y$ if $x \in O(y)$" is an equivalence relation on X.
(4) Prove that the orbit of x and the orbit of y either coincide exactly or are disjoint.

[1] Hint: Label the diagonals as $1, 2, 3, 4$. Note that every face has one vertex on each diagonal. For each face, list the diagonal of each vertex, conterclockwise, starting with 1. Note that each face has a different list.