Math 412. Group Actions.

1. Group Actions

Let $(G, \circ)$ be any group. Let X be any set (finite or infinite). An **action** of G on X is a natural way that the group elements "move around" the elements of X. Formally:

Definition 1.1. An *action* of G on X is a map

$$G \times X \to X$$
$$(g, x) \mapsto g \cdot x$$

which satisfies

(1) $e_G \cdot x = x$ for all $x \in X$.
(2) $g_1 \cdot (g_2 \cdot x) = (g_1 \circ g_2) \cdot x$ for all $g_1, g_2 \in G$ and all $x \in X$.

Here, it is important to absorb the correct use of notation: the set X has no extra structure (no operation) but the **action** allows us to combine a group element $g \in G$ with a set element $x \in X$ to get a new set element $g \cdot x \in X$. The two axioms of an action ensure that this procedure is compatible with the group structure. In particular, the first axiom tells us that the identity element of G behaves as expected: it does nothing to any $x \in X$.

Look carefully at Axiom 2: there are two different notations, $\cdot$ and $\circ$, and they mean two different things. Axiom 2 tells us that if we move an element $x \in X$ to another element of X using first g_2 and then g_1, the result is the same as if we let $g_1 \circ g_2$ act directly on x.

Example 1.2. Let G be the group $GL_2(\mathbb{R})$ of 2×2 matrices with real coefficients. Let $X = \mathbb{R}^2$ be the set of column vectors $\{\begin{bmatrix} x \\ y \end{bmatrix} \mid x, y \in \mathbb{R}\}$.

Then G acts on the set X in a natural way by left multiplication:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}.$$

It is easy to see that both axioms of a group action hold:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

and for any two matrices $A, B \in GL_2(\mathbb{R})$, we have

$$A \cdot (B \cdot \begin{bmatrix} x \\ y \end{bmatrix}) = A \cdot (B \begin{bmatrix} x \\ y \end{bmatrix}) = A(B \begin{bmatrix} x \\ y \end{bmatrix}) = (AB) \begin{bmatrix} x \\ y \end{bmatrix} = (AB) \cdot \begin{bmatrix} x \\ y \end{bmatrix},$$

by the associative law of matrix multiplication.

Example 1.3. Consider the dihedral group D_4, the symmetry group of the square. We can view D_4 as acting on the vertices of the square. If we label the vertices $\{1, 2, 3, 4\}$, say, in counterclockwise order from the top right, this gives an action of D_4 on the set $\{1, 2, 3, 4\}$. For example, the rotation r by 90^0 counterclockwise sends vertex 1 to vertex 2, where as the reflection d over the line through vertex 1 and vertex 3 will fix vertex 1. Likewise r sends vertex 2 to vertex 3, where as d will send vertex 2 to vertex 4. The composition $d \circ r$ will send vertex 1 first to vertex 2, then to vertex 4, so that

$$(d \circ r) \cdot \text{ vertex } 1 = d \cdot (r \cdot \text{ vertex } 1),$$

Example 1.4. Consider the group $G = \mathbb{Z} \times \mathbb{Z}$ under addition. This group acts on the Cartesian plane $\mathbb{R}^2$ in an obvious way:

$$(m, n) \cdot (x, y) = (m + x, n + y)$$

by translation. Observe that the identity element of the group, $(0, 0)$ leaves each element of $\mathbb{R}^2$ unchanged. So Axiom (1) of a group action holds. The element $(1, 0) \in G$, however, slides each element (x, y) of the Cartesian plane 1 unit to the right, to the point $(x + 1, y)$. To verify this is an action, we must also check Axiom (2), which in this example looks like

$$(m_1, n_1) \cdot [(m_2, n_2) \cdot (x, y)] = (m_1 + m_2, n_1 + n_2) \cdot (x, y),$$

using the fact that the operation of our group is $+$. It is easy check that this holds, by unraveling each side using the definition of the action. (Do it!)

2. The orbit of a point

Let G be a group acting a set X. Consider a point $x \in X$.

Definition: The **orbit** of x is the subset of X

$$O(x) := \{g \cdot x \,|\, g \in G\} \subset X.$$

Note that when a group G acts on a set X, *each point x of X* has an orbit—this orbit is a subset of X.

Example 2.1. Consider again Example 1.4 in which $\mathbb{Z}^2$ acts on the Cartesian plane $\mathbb{R}^2$ by translation. What is the orbit of the origin? We compute

$$O((0, 0)) = \{g \cdot (0, 0) \mid g \in \mathbb{Z}^2\} = \{(m, n) \in \mathbb{Z}^2\} = \mathbb{Z}^2.$$

Example 2.2. Let $\mathbb{R}^\times$ act on the Cartesian plane $\mathbb{R}^2$ by multiplication: for $\lambda \in \mathbb{R}^\times$ and $(x, y) \in \mathbb{R}^2$, define $\lambda \cdot (x, y) = (\lambda x, \lambda y)$. Let us compute some orbits:

(1) the orbit of $(0, 0)$ is the set $O((0, 0)) = \{\lambda \cdot (x, y) \mid \lambda \in \mathbb{R}^\times\} = \{(0, 0)\}$. That is, the origin is fixed by this action, and so makes a one-element orbit.

(2) The orbit of any non-zero (a, b) is the set

$$O((a, b)) := \{\lambda \cdot (a, b) \mid \lambda \neq 0\} = \{(\lambda a, \lambda b) \mid \lambda \in \mathbb{R}^\times\},$$

which we think of as the "punctured line" through the origin and (a, b)—here, the "puncture" means we omit the origin itself. Two non-zero points are in the same orbit if and only if they lie on the same line through the origin.

So for this action, we have two kinds of orbits: the origin, and the punctured lines through the origin. Note that $\mathbb{R}^2$ is the disjoint union of these orbits: every point lies in exactly one.

Example 2.3. Consider the matrices

$$\{\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}, \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}, \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}, \begin{bmatrix}0 & -1\\-1 & 0\end{bmatrix}, \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}\}.$$

It is not hard to check that these eight matrices form a subgroup of $GL_2(\mathbb{R})$. Let G act in the obvious way on vectors in $\mathbb{R}^2$. There are four different kinds of orbits, depending on how many of the coordinates are zero or equal:

(1) $O(\begin{bmatrix}0\\0\end{bmatrix}) = \{\begin{bmatrix}0\\0\end{bmatrix}\}$; This orbit has cardinality one. The origin is a **fixed point** of the action.

(2) $O(\begin{bmatrix} x \\ y \end{bmatrix})$ (where $x, y \neq 0$ and $x \neq y$) is

$$\{\begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} x \\ -y \end{bmatrix}, \begin{bmatrix} -x \\ y \end{bmatrix}, \begin{bmatrix} -x \\ -y \end{bmatrix}, \begin{bmatrix} y \\ x \end{bmatrix}, \begin{bmatrix} -y \\ x \end{bmatrix}, \begin{bmatrix} -y \\ -x \end{bmatrix}, \begin{bmatrix} y \\ -x \end{bmatrix}, \}$$

This orbit has cardinality eight, which is the largest we can ever expect for the action of an order eight group acting on a set. [Do you see why?]

(3) $O(\begin{bmatrix} 0 \\ y \end{bmatrix})$ where $y \neq 0$, is

$$\{\begin{bmatrix} 0 \\ y \end{bmatrix}, \begin{bmatrix} 0 \\ -y \end{bmatrix}, \begin{bmatrix} -y \\ 0 \end{bmatrix}, \begin{bmatrix} y \\ 0 \end{bmatrix}\}$$

Note that this orbit is the same as $O(\begin{bmatrix} y \\ 0 \end{bmatrix})$. These orbits are cardinality four.

(4) $O(\begin{bmatrix} x \\ x \end{bmatrix})$, where $x \neq 0$, is

$$\{\begin{bmatrix} x \\ x \end{bmatrix}, \begin{bmatrix} -x \\ -x \end{bmatrix}, \begin{bmatrix} x \\ -x \end{bmatrix}, \begin{bmatrix} -x \\ x \end{bmatrix}\}$$

Note that this orbit is the same as the orbit of the point $\begin{bmatrix} -x \\ x \end{bmatrix}$. Again, this orbit has cardinality 4.

THE PARTITION OF X INTO ORBITS. Suppose that a group G acts on a set X. Then we can define an equivalence relation on X as follows: say "x is equivalent to y" (and write $x \sim y$) if $x \in O(y)$. This is an equivalence relation because

(1) $x \in O(x)$ for all $x \in X$;
(2) $x \in O(y)$ if and only if $y \in O(x)$;
(3) $x \in O(y)$ and $y \in O(z)$ implies that $x \in O(z)$.

Proof. Proof that $\sim$ is an equivalence relation There are three things to show:

(1) $x \sim x$, which is trivial since $x = e \cdot x \in O(x)$.
(2) $x \sim y$ implies $y \sim x$, which is easy. Indeed, if $x \in O(y)$, we can write $x = g \cdot y$, so that $g^{-1}x = y$. This shows that $y \in O(x)$.
(3) $x \sim y$ and $y \sim z$ implies $x \sim z$, which is also easy. If $x \in O(y)$ and $y \in O(z)$, we can write $x = g \cdot y$ and $y = h \cdot z$ for some $g, h \in G$. So $x = g \cdot y = g \cdot (h \cdot z) = (gh) \cdot z$, showing $x \in O(z)$.

$\square$

Proposition 2.4. *Suppose that a group G acts on a set X. Let x and y be two points of X. Then either*

$$O(x) = O(y) \qquad \textit{or} \qquad O(x) \cap O(y) = \emptyset.$$

In particular, every element of X belongs to **exactly one** *orbit.*

Proof. To prove the statement, it is enough to show that if $O(x) \cap O(y)$ is non-empty, then then $O(x) = O(y)$.

Assume $w \in O(x) \cap O(y)$. Then in particular, $w \in O(x)$. So we can write $w = h \cdot x$ for some $h \in G$. But then also for any $g \in G$, we have $g \cdot w = g \cdot (h \cdot x) = (g \circ h) \cdot x$. This shows that $O(w) \subset O(x)$. But since $w = h \cdot x$ implies also that, $h^{-1} \cdot w = x$, the same argument shows that $O(x) \subset O(w)$. So $O(x) = O(w)$. Repeating this argument starting from $w \in \mathcal{O}(y)$, we also get that $O(w) = O(y)$. So the sets $O(x)$ and $O(y)$ must be equal, as needed.

It follows that every element of X is in **exactly one** orbit. For if there is some $w \in X$ which in in two different orbits, $O(x) \neq O(y)$, then we immediately get a contradiction: $O(x) \cap O(y)$ is empty so it cannot contain w.

$\square$

The way to think about Proposition 2.4 is that the action of G on X will *partition up* X into non-overlapping orbits. The group G moves points around within each orbit: you can get from any point in one orbit to any other point in the same orbit by letting G act. But G can never move a point from one orbit into a *different* orbit.

Example 2.5. Consider the rotation group $SO_2(\mathbb{R}) = \{\begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix} \mid \theta \in [0, 2\pi)\}$. It acts on the plane $\mathbb{R}^2$ in an obvious way:

$\begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix} \cdot \begin{bmatrix}x \\ y\end{bmatrix} = \begin{bmatrix}x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta\end{bmatrix} =$ the rotation of the vector $\begin{bmatrix}x \\ y\end{bmatrix}$ through an angle of θ counter-clockwise.

Let us compute the orbits. First note that for any $A \in SO_2(\mathbb{R})$, we have $A\begin{bmatrix}0 \\ 0\end{bmatrix} = \begin{bmatrix}0 \\ 0\end{bmatrix}$. So

$$O(\begin{bmatrix}0 \\ 0\end{bmatrix}) = \{\begin{bmatrix}0 \\ 0\end{bmatrix}\},$$

and the orbit of the origin contains only itself. We say that the origin is **fixed point** of this action, since every element of the group leaves it fixed.

Now take any point $p = \begin{bmatrix}a \\ b\end{bmatrix}$ other than $\begin{bmatrix}0 \\ 0\end{bmatrix}$. Applying a rotation $A \in SO_2$, we get another point on the circle with center $\begin{bmatrix}0 \\ 0\end{bmatrix}$ though p. As we apply *all* the elements of SO_2, we get *all* the points on this circle. That is,

$$O(\begin{bmatrix}a \\ b\end{bmatrix}) = \text{circle centered at origin through } \begin{bmatrix}a \\ b\end{bmatrix}.$$

The orbit of p is the circle centered at the origin through p.

The space $\mathbb{R}^2$ is partitioned up into orbits: each circle centered at the origin (including the trivial "circle" of radius zero) is one orbit. As we range through all the different size circles, these circles cover the whole of $\mathbb{R}^2$. Every point of $\mathbb{R}^2$ is in *exactly one* orbit. So we see that $\mathbb{R}^2$ is the disjoint union of the orbits for this action.

Example 2.6. Consider the natural action of S_5 on the set of $X = \{1, 2, 3, 4, 5\}$. That is, $\sigma \in S_5$ acts on $i \in \{1, 2, 3, 4, 5\}$ by simply $\sigma(i)$. You should check that this is an action.

What is the orbit of the point $1 \in X$? Well, the transposition $\tau_i = (1\,i)$ sends $1 \mapsto i$. So basically, we can send 1 to any element of X with this action. This means $O(1) = \{1, 2, 3, 4, 5\}$, the whole set. Likewise, this is also the orbit of 2, or any element of X. All the orbits are the same: $O(1) = O(2) = O(3) = O(4) = O(5) = X$. There is only one orbit, and X is the trivial union of its distinct orbit(s).

Example 2.7. Consider the natural action of S_n on the set of $\mathcal{P}$ on subsets of $\{1, 2, 3, \dots, n\}$. That is, $\sigma \in S_n$ acts on a subset $Y = \{i_1, i_2 \dots, i_t\} \subset \{1, 2, \dots, n\}$ by simply $\sigma(Y) = \{\sigma(i_1), \sigma(i_2) \dots, \sigma(i_t)\} \subset \{1, 2, \dots, n\}$. You should check that this is an action of S_n on $\mathcal{P}$.

Let us compute the orbit of the subset $\{1\} \in \mathcal{P}$. Each $\sigma \in S_n$ will take the set $\{1\}$ to the one element set $\{\sigma(1)\}$, so the orbit contains only one element sets. But in fact the orbit consists of *all* one-element sets, since the transposition $\tau_i = (1\,i)$ sends $\{1\} \mapsto \{i\}$.

Likewise, the orbit of $\{1, 2\}$ is the set of all two-element sets. To see this, assume that $j \neq 1, 2$. Consider an arbitrary two-element subset $\{i, j\}$. If $\{i, j\}$ =$\{1, 2\}$, then it is clearly in the orbit, since $e \cdot \{1,2\} = \{1,2\}$. So we can assume without loss of generality that $j \neq 1,2$. To see that $\{i,j\}$ is in the orbit of $\{1,2\}$, we observe that there is a $\sigma \in S_n$ which takes 1 to i and 2 to j (for example, $\sigma = (1\ i)(2\ j)$ has this property). Applying σ to $\{1,2\}$, we compute: $\sigma \cdot \{1,2\} = \{i,j\}$. Thus the orbit $\{1,2\}$ is the set of *all* 2-element subsets of $\{1,2,\ldots,n\}$.

In a similar way, we see that the set of all 3-element subsets of $\{1,2,3,\ldots,n\}$ forms one orbit in $\mathcal{P}$. And the set of all four-element subsets of $\{1,2,3,\ldots,n\}$ forms a different orbit of $\mathcal{P}$, etc.

The partition of $\mathcal{P}$ into orbits for this action is the same as the partition of $\mathcal{P}$ into collections of sets of the same cardinality. Clearly there is a total of $n+1$ orbits in this partition of $\mathcal{P}$, one for each possible cardinalty $0,1,\ldots,n$. The empty set (cardinality zero) and the whole set $\{1,2,\ldots,n\}$ (cardinality n) are fixed points of the action of S_n on $\mathcal{P}$: each of their orbits consists of only one element of $\mathcal{P}$: the orbit of $\emptyset$ is $\{\emptyset\}$ and the orbit of $\{1,2,\ldots,n\}$ is $\{\,\{1,2,\ldots,n\}\,\}$.

3. Stabilizers and the Orbit Stabilizer theorem

Let G be a group acting a set X. Consider a point $x \in X$.

Definition 3.1. The **stabilizer** of x is the subset of G

$$Stab(x) = \{g \in G \mid g \cdot x = x\}.$$

Each $x \in X$ has an **orbit** under the group action, which is a subset of X, and a **stabilizer**, which is a subset of G. Do not confuse the two concepts! The stabilizer of a point x, however, has additional structure:

Proposition 3.2. *Suppose a group G acts on a set X. Fix any $x \in X$. Then the stabilizer of x*

$$Stab(x) = \{g \in G \ \mid \ g \cdot x = x\}$$

is a subgroup of G.

Proof. There are three things to check:

(1) $Stab(x)$ is non-empty. This is clear, since $e_G \cdot x = x$, so that $e_G \in Stab(x)$.
(2) $Stab(x)$ is closed under the operation of G: for this, take any $g,h \in Stab(x)$. We need to verify that $g \circ h \in Stab(x)$. This follows from Axiom 2, since $(g \circ h) \cdot x = g \cdot (h \cdot x) = g \cdot x = x$.
(3) $Stab(x)$ is closed under taking inverses. For this, take $g \in Stab(x)$. We need to show that $g^{-1} \in Stab(x)$. Since $g \cdot x = x$, we can apply g^{-1} to both sides to get $g^{-1} \cdot (g \cdot x) = g^{-1} \cdot x$. Now using the axioms of a group action to simplify the left hand sids, we have that $x = g^{-1} \cdot x$. So $g^{-1} \in Stab(x)$.

Since $Stab(x)$ non-empty, closed under the operation from G and closed under taking inverse in G. it is a subgroup of G.

□

Example 3.3. Consider again the group

$$G = \{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \}$$

of Example 2.3. We have already computed its orbits for its natural action on $\mathbb{R}^2$. Let's compute some stabilizers. Every element of G fixes the origin, so

$$stab(\begin{bmatrix} 0 \\ 0 \end{bmatrix}) = G.$$

Now consider a point of the form $\begin{bmatrix} 0 \\ y \end{bmatrix}$, where $y \neq 0$. Of course the identity fixes it, but also the matrix $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. These are the only two matrices which fix this point, so

$$stab(\begin{bmatrix} 0 \\ y \end{bmatrix}) = \{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\},$$

which is a cyclic group of order 2. Similarly, points of the form $\begin{bmatrix} x \\ 0 \end{bmatrix}$ where $x \neq 0$ have stabilizer

$$stab(\begin{bmatrix} x \\ 0 \end{bmatrix}) = \{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\}.$$

It is also easy to check that (when $x \neq 0$),

$$stab(\begin{bmatrix} x \\ x \end{bmatrix}) = \{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\},$$

and that for $x \neq y \neq 0$,

$$stab(\begin{bmatrix} x \\ y \end{bmatrix}) = \{e\},$$

Observe that in this example that points in the same orbit may have different stabilizers, but the order of the stabilizers for all x in the same orbit is the same. This is not an accident! It follows from the next theorem, which is a widely useful tool for understanding groups and the many ways that they can act on sets.

Theorem 3.4. *If a finite group G acts on a set X, then for every $x \in X$, we have*

$$|G| = |O(x)| \times |Stab(x)|.$$

Proof. Fix $x \in X$. There is a surjective map $G \to O(x)$ sending $g \mapsto g \cdot x$. Let $K \subset G$ be the stabilizer of x. We claim that this map induces a well-defined bijection from the set G/K of left K-cosets of G to $O(x)$:

$$G/K \to O(x) \quad gK \mapsto g \cdot x.$$

To check this, we need to make sure that every element in the left coset gK goes the same element $g \cdot x$, to be sure it is well-defined. But for $k \in K = Stab(x)$, we have $(gk) \cdot x = g \cdot (k \cdot x) = g \cdot x$. For surjectivity, note that $g \in G$ is in some coset, so every $g \cdot x$ is in the image. To check injectivity, suppose that gK and hK have the same image. This means that $g \cdot x = h \cdot x$, so that $h^{-1}g \in Stab(x) = K$. We conclude that $g \in hK$, so that $gK = hK$. This shows that there is a bijection between the set of all left cosets of $Stab(x)$ and the orbit $O(x)$. By Lagrange's theorem, the number of left cosets is $[G : K] = |G|/|K|$. So $|G| = |K| \times |O(x)|$. $\square$

Example 3.5. Let G be the rotational symmetry group of the cube. Consider the action of G on the set of the six faces of the cube. Fix one face F. Since we can rotate to move any face to any other, the orbit of F is the full set of all six faces. On the other hand, the rotations that fix F are the four rotations (including the identity) around the axis perpendicular to F. Thus $Stab(F)$ is a cyclic group of order 4. In particular

$$|G| = |O(F)| \times |Stab(F)| = 6 \times 4 = 24.$$

4. A DIFFERENT WAY TO THINK ABOUT GROUP ACTIONS

Suppose that a group G acts on a set X. One way to understand this is to imagine picking one $g \in G$ and thinking about where *that* g sends each element of X. Each element of G gives rise to a mapping

$$X \xrightarrow{\phi_g} X \qquad x \mapsto g \cdot x.$$

This mapping is a bijection, because it has inverse

$$X \xrightarrow{\phi_{g^{-1}}} X \qquad x \mapsto g^{-1} \cdot x.$$

So each element of G determines a *bijection* of X to itself. That is, each ϕ_g is an element of the set Bij(X) of all bijections from X to itself. Putting these together, we have a mapping

$$G \longrightarrow \text{Bij}(X)$$
$$g \mapsto \phi_g.$$

This map is sometimes called the **adjunction map** determined by the group action.

Recall that Bij(X) is always a group under composition. So the adjunction map is a map between two groups, and it is natural to wonder whether it is a group homomorphism. In fact, the axioms of a group action guarantee this!

Formally:

Theorem 4.1. *Let G be a group acting a set X. Then there is an induced group homomorphism*

$$G \xrightarrow{\Phi} \text{Bij}(X) \qquad g \mapsto [X \xrightarrow{\phi_g} X \quad x \mapsto g \cdot x],$$

where Bij(X) denotes the group of all bijections from X to itself, under composition. Conversely, given a group homomorphism

$$G \xrightarrow{\Phi} \text{Bij}(X)$$

we can recover a group action of G on X as follows: $g \cdot x = \Phi(g)(x)$.

Proof. Fix an action of a group G on a set X. We need to prove that the adjunction map is a **group homomorphism.** That is, we must show that for any $g, h \in G$, $\Phi(gh) = \Phi(g) \circ \Phi(h)$. By definition of Φ, this says we must show that $\phi_{gh} = \phi_g \circ \phi_h$. These are two different bijections $X \to X$, so to show that they are equal, we must show that for every input $x \in X$, they have the same output.

By definition, $\phi_{gh}(x) = (gh) \cdot x$. On the other hand $\phi_g \circ \phi_h(x) = g \cdot (h \cdot x)$. So we must verify that $(gh) \cdot x = g \cdot (h \cdot x)$. But this is one of the axioms of a group action! Thus Φ is a group homomorphism.

For the other direction, assume that Φ is a group homomorphism. Then $\Phi(e)$ is the identity in Bij(X). This means that ϕ_e does nothing to any x, or in other words $e \cdot x = x$ for all $x \in X$. This verifies the first axiom of a group action.

For the second axiom, we use that $\Phi(gh) = \Phi(g) \circ \Phi(h)$. This means that $\phi_{gh} = \phi_g \circ \phi_h$, which again means that $(gh) \cdot x = g \cdot (h \cdot x)$ for all $x \in X$. The second axiom is verified. QED.

□

Example 4.2. As an application of the adjunction mapping, we can better understand the rotational symmetry group G of the cube. Note that G acts naturally on the set of 4 grand diagonals of the cube. The action of G on this four-element set is equivalent to a group homomorphism

$$G \to \mathcal{S}_4.$$

Since both groups G and S_4 have 24 elements, this map will be an isomorphism if it is injective (or surjective). To check that it is injective, consider an element $g \in G$ is in the kernel. This means g induces trivial bijection— the identity map—on the set of grand diagonals. In other words, g is in the stabilizer of each grand diagonal. But the stabilizer of each grand diagonal is an order three group of rotations around that diagonal. The intersection of any two of these stabilizers is $\{e_G\}$ (since the intersection is a subgroup of both), so the entire kernel is trivial. This proves that $G \cong S_4$. The symmetry group of a cube is isomorphic to S_4!