Homework #11

Problems to hand in on Thursday, April 18, in the beginning of class. Write your answers out carefully, staple pages, and write your name and section number on each page.

1) Consider the group $G = \mathbb{Z}_{40}^{\times}$.
   (a) G has 16 elements. List them.
   (b) The subgroup $H = \langle [7] \rangle$ has four elements. List the four elements of G/H. Note that each element of G/H is a set; for each of these sets, list all of its elements.
   (c) Use the definition of the group structure of G/H to create a multiplication table for the quotient group G/H.

2) Let G be an abelian group, not necessarily finite.
   (a) Show that the set T of elements of G of finite order forms a subgroup of G.
   (b) Show that every nonidentity element of G/T has infinite order.

3) Let $O(2)$ denote the subgroup of *orthogonal* 2×2 matrices in $M_2(\mathbb{R})$.[1]
   (a) Compute the kernel and the image of the determinant homomorphism $\det\colon O(2) \to \mathbb{R}^{\times}$.
   (b) Use part (a) to show that $O(2)/SO(2) \cong \{\pm 1\}$. Describe the elements of $O(2)/SO(2)$: what sets of linear transformations from $\mathbb{R}^2 \to \mathbb{R}^2$ are they?
   (c) Find two elements $M, N \in O(2)$ of finite order whose product has infinite order. Conclude that the set of elements of finite order in $O(2)$ do not form a subgroup.

THEOREM 9.7: FUNDAMENTAL STRUCTURE THEOREM FOR FINITE ABELIAN GROUPS: Let G be a finite abelian group. Then G is isomorphic to a group of the form

$$\mathbb{Z}_{p_1^{a_1}} \times \mathbb{Z}_{p_2^{a_2}} \times \mathbb{Z}_{p_3^{a_3}} \times \cdots \times \mathbb{Z}_{p_n^{a_n}}$$

where $p_1, p_2, \ldots p_n$ are (not necessarily distinct!) prime numbers. Moreover, the product is unique, up to re-ordering the factors.

4) (a) Suppose that G is abelian and has order 8. Use the Structure Theorem for Finite Abelian Groups to show that up to isomorphism, G must be isomorphic to one of three possible groups, each a product of cyclic groups of prime power order.
   (b) Determine the number of abelian groups of order 12, up to isomorphism.
   (c) For p prime, how many isomorphism types of abelian groups of order p^5?
   (d) If an abelian group of order 100 has no element of order 4, prove that G contains a Klein 4-group.

[1] If you aren't familiar with this notion from 217, it means matrices $M = [\,\vec{v}\,\vec{w}\,]$ with $\vec{v} \cdot \vec{v} = \vec{w} \cdot \vec{w} = 1$ and $\vec{v} \cdot \vec{w} = 0$.