Homework #11

Problems to hand in on Thursday, April 18, in the beginning of class. Write your answers out carefully, staple pages, and write your name and section number on each page.

1) Consider the group $G = \mathbb{Z}_{40}^{\times}$.

(a) G has 16 elements. List them.

(b) The subgroup $H = \langle [7] \rangle$ has four elements. List the four elements of G/H. Note that each element of G/H is a set; for each of these sets, list all of its elements.

(c) Use the definition of the group structure of G/H to create a multiplication table for the quotient group G/H.

Solution.

(a) We will drop the brackets for ease of notation.
$\{1, 3, 7, 9, 11, 13, 17, 19, 21, 23, 27, 29, 31, 33, 37, 39\}$.

(b) The elements of G/H are the four cosets:

$$H = \{1, 7, 9, 23\}$$
$$3H = \{3, 21, 27, 29\}$$
$$11H = \{11, 13, 19, 37\}$$
$$17H = \{17, 31, 33, 39\}$$

(c)

	H	$3H$	$11H$	$17H$
H	H	$3H$	$11H$	$17H$
$3H$	$3H$	H	$17H$	$11H$
$11H$	$11H$	$17H$	H	$3H$
$17H$	$17H$	$11H$	$3H$	H

2) Let G be an abelian group, not necessarily finite.

(a) Show that the set T of elements of G of finite order forms a subgroup of G.

(b) Show that every nonidentity element of G/T has infinite order.

Solution.

(a) We note first that $e \in T$. If $t, t' \in T$, then there are integers m, n such that $t^m = (t')n = e$. Then $(tt')^{mn} = t^{mn}t'^{mn}$, using the abelian property, and $t^{mn} = (t^m)^n = e$, and $t'^{mn} = ((t')^n)^m = e$, so $(tt')^{mn} = e$. Finally, the order of an element is equal to the order of its inverse, so T is closed under inverses. It follows that T is a subgroup.

(b) Let $Tg \in G/T$. If g has finite order in G then $g \in T$ so $Tg = T$, which is the identity element. If $Tg \neq T$, then $g \notin T$. Suppose that $(Tg)^n = T$. Then $g^n \in T$, so for some m, $e = (g^n)^m = g^{mn}$, so $g \in T$. Thus, the only element of finite order is the identity.

3) Let $O(2)$ denote the subgroup of *orthogonal* 2×2 matrices in $M_2(\mathbb{R})$.[1]

(a) Compute the kernel and the image of the determinant homomorphism $\det\colon O(2) \to \mathbb{R}^\times$.

(b) Use part (a) to show that $O(2)/SO(2) \cong \{\pm 1\}$. Describe the elements of $O(2)/SO(2)$: what sets of linear transformations from $\mathbb{R}^2 \to \mathbb{R}^2$ are they?

(c) Find two elements $M, N \in O(2)$ of finite order whose product has infinite order. Conclude that the set of elements of finite order in $O(2)$ do not form a subgroup.

Solution.

(a) The kernel is the set of orthogonal matrices with determinant 1. We called this $SO(2)$ in a previous assignment. Every orthogonal matrix has determinant ± 1, so the image is the group $\{\pm 1\}$.

(b) If we change the target of the determinant and consider the surjective homomorphism $\det\colon O(2) \to \{\pm 1\}$, the First Isomorphism Theorem says that $O(2)/SO(2) \cong \{\pm 1\}$. The two cosets are $SO(2)$, the set of rotations of the plane, and $O(2) \setminus SO(2)$, the set of reflections of the plane.

(c) Let α be a number such that $\frac{\alpha}{2\pi}$ is irrational. Let $M = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ be the matrix for the linear transformation of reflection over the x-axis, and let $N = \begin{bmatrix} \cos(\alpha) & \sin(\alpha) \\ \sin(\alpha) & -\cos(\alpha) \end{bmatrix}$ be the matrix for reflection over the line that makes the angle $\frac{\alpha}{2}$ with the x-axis. These reflections have order 2. The product MN is rotation by α, which has infinite order.

Theorem 9.7: Fundamental Structure Theorem for Finite Abelian Groups: Let G be a finite abelian group. Then G is isomorphic to a group of the form

$$\mathbb{Z}_{p_1^{a_1}} \times \mathbb{Z}_{p_2^{a_2}} \times \mathbb{Z}_{p_3^{a_3}} \times \cdots \times \mathbb{Z}_{p_n^{a_n}}$$

where $p_1, p_2, \ldots p_n$ are (not necessarily distinct!) prime numbers. Moreover, the product is unique, up to re-ordering the factors.

4) (a) Suppose that G is abelian and has order 8. Use the Structure Theorem for Finite Abelian Groups to show that up to isomorphism, G must be isomorphic to one of three possible groups, each a product of cyclic groups of prime power order.

(b) Determine the number of abelian groups of order 12, up to isomorphism.

(c) For p prime, how many isomorphism types of abelian groups of order p^5?

(d) If an abelian group of order 100 has no element of order 4, prove that G contains a Klein 4-group.

Solution.

[1] If you aren't familiar with this notion from 217, it means matrices $M = [\,\vec{v}\,\vec{w}\,]$ with $\vec{v}\cdot\vec{v} = \vec{w}\cdot\vec{w} = 1$ and $\vec{v}\cdot\vec{w} = 0$.

(a) By the Structure Theorem, G is isomorphic to either $\mathbb{Z}_8$, $\mathbb{Z}_4 \times \mathbb{Z}_2$, or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

(b) There are 2: $\mathbb{Z}_3 \times \mathbb{Z}_4$ and $\mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

(c) We just need to count all the ways to write p^5 as a product of prime powers: $p^5,\ p^4 \times p,\ p^3 \times p^2,\ p^3 \times p \times p,\ p^2 \times p^2 \times p,\ p^2 \times p \times p \times p,\ p \times p \times p \times p \times p$. So there are 7 abelian groups of order p^5 (up to isomorphism).

(d) An abelian group of order 100 that does not contain an element of order 4 must be isomorphic to either $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{25}$, or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5 \times \mathbb{Z}_5$. The first contains the Klein 4-group $\{(a, b, 0) \mid a, b \in \mathbb{Z}_2\}$, and the second contains the Klein 4-group $\{(a, b, 0, 0) \mid a, b \in \mathbb{Z}_2\}$.