Homework #4

Problems to hand in on Thursday, February 14, in the beginning of class. Write your answers out carefully, staple pages, and write your name and section number on each page.

1) Let m and n be positive integers with $(m, n) = 1$. Show[1] that $\mathbb{Z}_{mn} \cong \mathbb{Z}_m \times \mathbb{Z}_n$.

2) Let V be a vector space. Recall that a function $T : V \to V$ is a *linear transformation* if for all $v, w \in V$ and all $\lambda \in \mathbb{R}$, we have $T(v + w) = T(v) + T(w)$ and $T(\lambda v) = \lambda T(v)$.

   (a) Show that the set of linear transformations from V to V, with usual addition, and *composition of functions* as multiplication, forms a ring.

   (b) Consider the vector space $\mathbb{R}[x]$ and let $\mathcal{L}(\mathbb{R}[x])$ be the ring of linear transformations of $\mathbb{R}[x]$ as defined in the previous part. Consider the element $\frac{d}{dx} \in \mathcal{L}(\mathbb{R}[x])$. Show that there is an element $F \in \mathcal{L}(\mathbb{R}[x])$ such that $\frac{d}{dx}F = 1_{\mathcal{L}(\mathbb{R}[x])}$, but there is no element $G \in \mathcal{L}(\mathbb{R}[x])$ such that $G\frac{d}{dx} = 1_{\mathcal{L}(\mathbb{R}[x])}$.

3) We say a ring R has characteristic n if n is the smallest positive integer such that

$$\underbrace{1 + \cdots + 1}_{\text{n times}} = 0.$$

   If no such n exists, we say that R has characteristic 0.

   (a) Give examples of a ring of characteristic 0 and a ring of characteristic n for each $n \geqslant 2$.

   (b) Suppose that R is a commutative ring of prime characteristic p. Prove that the *Freshman's Dream* holds in R: $(a + b)^p = a^p + b^p$ for all $a, b \in R$.

   (c) Suppose that R is a commutative ring of prime characteristic p. Prove that the Frobenius map $r \mapsto r^p$ is a ring homomorphism $R \longrightarrow R$.

   (d) Give an example to show that if the characteristic of R is not 2, $r \mapsto r^2$ may not be a ring homomorphism.

4) Consider the field $\mathbb{F} = \mathbb{Z}_{13}$. Construct the addition and the multiplication tables for this field and use them to answer the following questions.

   (a) Give a reasonable interpretation, in $\mathbb{F}$, for the expressions $2, -4, 3/4, -4/3, \sqrt{-1}$ (and carefully explain your reasoning).

   (b) Solve the quadratic equation $x^2 + 6x + 4 = -1$ by *completing the square*. Check your answers!

   (c) Now solve the same equation by using the *quadratic formula*. Why is it valid over $\mathbb{F}$? Is it valid over any field?

   (d) Use the usual discriminant $D = b^2 - 4ac$ to classify the equations $ax^2 + bx + c = 0$ that have two roots, a single root, or no root in $\mathbb{F}$.

   (e) Using the discriminant determine, without solving the equation, the number of roots of the equation $7x^2 + 4x + 3 = 0$.

[1] Hint: You can save a lot of work by referring back to problems from previous homeworks.