Homework #5

Problems to hand in on Thursday, February 21, in the beginning of class. Write your answers out carefully, staple pages, and write your name and section number on each page.

1) Let $R = \mathbb{R}[x, y]$ be the polynomial ring in two variables x, y.

   a) Let $S \subseteq \mathbb{R}^2$. Show that the set
   $$I_S := \{f(x, y) \in R \mid f(a, b) = 0 \text{ for all } (a, b) \in S\}$$
   is an ideal of $\mathbb{R}[x, y]$.

   b) If $S = \{(0, 0)\}$, what is I_S?

   c) Show that, with I_S as in b), $R/I_S \cong \mathbb{R}$.

2) Consider the ring $M_2(\mathbb{R})$.

   a) Take any nonzero 2×2 matrix A. Show that by multiplying A on the left by matrices of the form
   $$\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}, \begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$
   we can do any elementary row operation to A.

   b) State a way of interpreting column operations using matrix multiplication.

   c) Prove that the only ideals in $M_2(\mathbb{R})$ are $\{0\}$ and $M_2(\mathbb{R})$.

3) Let $\mathbb{F}$ be a field, and $R = \mathbb{F}[x]$ be a polynomial ring. For $f(x) \in R$, we say that two polynomials $g(x), h(x)$ are **congruent modulo** $f(x)$ if $f(x)|(g(x) - h(x))$. We write $g(x) \equiv h(x) \bmod f(x)$ to denote this.[1]

   a) Show that for $\lambda \in \mathbb{F}$, and any $a(x) \in R$, $a(x) \equiv a(\lambda) \bmod x - \lambda$.

   b) Suppose that the greatest common divisor of $b(x)$ and $c(x)$ is 1. Show that for *any* $d(x)$, there is a polynomial $e(x)$ that solves the congruence equation $c(x)e(x) \equiv d(x) \bmod b(x)$.

   c) Suppose again that the greatest common divisor of $f(x)$ and $g(x)$ is 1. Show that for *any* $h(x)$ and $i(x)$, there is a polynomial $j(x)$ that solves the system of congruence equations
   $$\begin{cases} j(x) \equiv h(x) \bmod f(x) \\ j(x) \equiv i(x) \bmod g(x). \end{cases}$$

   d) Let $\alpha, \beta, \gamma, \delta \in \mathbb{F}$, with $\alpha \neq \beta$. Use the previous parts to show that there is a polynomial $k(x) \in R$ such that $k(\alpha) = \gamma$ and $k(\beta) = \delta$.

4) Let $S \subset \mathbb{Q}$ be the subset of rational numbers with odd denominators (when expressed in lowest terms).

   (a) Show that S is a subring of $\mathbb{Q}$.

   (b) Let $I \subseteq S$ be the subset of rational numbers with even numerator (when expressed in lowest terms). Prove that I is an ideal of S.

   (c) Show that the quotient ring S/I is isomorphic to $\mathbb{Z}_2$.

[1] This is a special case of congruence modulo an ideal; namely congruence modulo the ideal $(f(x))$.