Homework #6

Problems to hand in on Thursday, March 14, in the beginning of class. Write your answers out carefully, staple pages, and write your name and section number on each page.

1) Let R be a commutative ring. Recall that $e \in R$ is an idempotent if $e^2 = e$.

   a) Show that if e is an idempotent, then so is $1 - e$.

> Fact: If $e \neq 0, 1$ is an idempotent, then the ideal generated by e is a ring of its own, with the same multiplication and addition structure as R, but with a *different* multiplicative identity: e. We write Re to represent this ring.

   b) Show that the map

$$\begin{aligned} R &\xrightarrow{\varphi} Re \times R(1-e) \\ r &\longrightarrow (re, r - re) \end{aligned}$$

   is a ring isomorphism.

   c) Show that a ring R is isomorphic to a direct product of two nonzero rings if and only if R contains an idempotent element other than 0_R and 1_R.

2) Recall: an ideal $P \neq R$ in a ring R is prime if $ab \in P$ implies $a \in P$ or $b \in P$.

   a) Prove that P is prime if and only if R/P is a domain.

   b) Use the first isomorphism theorem to show that the ideals (x) and $(2, x)$ in $\mathbb{Z}[x]$ are prime ideals.[1][2]

   c) Show that the ideal $(4, x)$ in $\mathbb{Z}[x]$ is not prime.

   d) Show that the ideal $(2, \sqrt{10})$ in $\mathbb{Z}[\sqrt{10}] = \{a + b\sqrt{10} \mid a, b \in \mathbb{Z}\} \subseteq \mathbb{R}$ is prime.

   e) Is the ideal (2) in $\mathbb{Z}[i]$ a prime ideal?

[1] Hint: For the first one, consider the homomorphism $\mathbb{Z}[x] \to \mathbb{Z}$ "evaluate at zero".
[2] Reminder: $(2, x)$ refers to the ideal generated by 2 and x.