Math 412. Review questions

- Definition of ring
  - What is an operation?
  - What does it mean to be associative?
  - What does it mean to be commutative?
  - What does it mean to have an identity?
  - What does it mean for an element to have an inverse?
  - What does it mean for two operations to satisfy distributive laws?
  - What is the definition of a ring?[1]
  - What is the definition of a subring?
  - How do you define subtraction in a ring?
  - Can you use the axioms to prove that $(-a)(-b) = ab$, and other basic things?
- Examples/constructions of rings
  - Which familiar sets of numbers are rings?
  - Is $\mathbb{Z}_N$ a ring? What are the operations?
  - Why is $R \times S$ a ring if R, S are? What the the operations, and 0 and 1?
  - How do you check if a subset of a ring is a subring?
  - Do you know rings where the multiplication is not commutative?
  - Are there infinite rings with finite subrings?
  - Are there commutative rings with noncommutative subrings?
  - Are there noncommutative rings with commutative subrings?
- Special types of rings/elements
  - When does $0 = 1$ in a ring?
  - What is a commutative ring?
  - What is an (integral) domain?
  - What is a field?
  - Which of the last few notions imply each other? Why? What if the ring is finite?
  - Can you cancel addition in a ring?
  - Can you cancel multiplication by a nonzero element in a ring? If not, can you do it in one of the special types of rings above?
  - What is a zerodivisor? What does it have to do with these special ring types?
  - What is a unit? What does it have to do with these special ring types?
  - What is a nilpotent?
  - What is an idempotent?
  - In what type of ring do you have all four operations $+, -, \times, \div$ (except dividing by zero)? How is division defined in such a ring?
- Homomorphisms
  - What is a homomorphism?
  - What is an isomorphism?
  - Can you find homomorphisms that are injective but not surjective? Surjective but not injective? Neither?
  - What is the kernel of a homomorphism?
  - What is the image of a homomorphism?
  - What special property does the kernel have?
  - What special property does the image have?
  - When is there a homomorphism $\mathbb{Z}_N \to \mathbb{Z}_M$?

[1] A ring always has a multiplicative identity in this class.

  - Given a ring R, what ring homomorphisms $\mathbb{Z} \longrightarrow R$ are there?
- Ideals
  - What is an ideal?
  - How do you check a subset of a ring is an ideal?
  - Are ideals subrings? Are subrings ideals?
  - What two ideals does every ring have?
  - How is checking a subset is an ideal different in a commutative ring vs a noncommutative one?
  - What is the ideal generated by $a_1, \ldots, a_t$ in a commutative ring?
  - What are generators of an ideal?
  - What is congruence modulo an ideal?
  - Must every ideal I in a ring R be the kernel of some ring homomorphism $R \longrightarrow S$?
- $\mathbb{Z}$
  - What is the division algorithm in $\mathbb{Z}$?
  - What is the Euclidean algorithm in $\mathbb{Z}$?
  - What is the fundamental theorem of arithmetic in $\mathbb{Z}$?
  - What are the units in $\mathbb{Z}$?
  - What are the zerodivisors in $\mathbb{Z}$?
  - What is the GCD of two elements in $\mathbb{Z}$?
  - What is a prime in $\mathbb{Z}$?
  - What special property do primes have for dividing other numbers?
  - Is the GCD of two integers a linear combination? How do you find it as such?
  - What are the ideals in $\mathbb{Z}$?
  - What rings can we obtain as quotients of $\mathbb{Z}$?
- $\mathbb{Z}_N$
  - What is $\mathbb{Z}_N$? What are its elements?
  - What are the units in $\mathbb{Z}_N$?
  - What are the zerodivisors in $\mathbb{Z}_N$?
  - When is $\mathbb{Z}_N$ a field? A domain?
  - For which N does $ax = b$ always have a solution in $\mathbb{Z}_N$, if $a \neq 0$?
  - How do you find inverses in $\mathbb{Z}_N$?
  - How do you solve $ax = b$ in $\mathbb{Z}_N$, when it does have a solution?
  - What are the ideals in $\mathbb{Z}_N$?
  - What does the Chinese remainder theorem say?
- $\mathbb{F}[x]$
  - What is the division algorithm in $\mathbb{F}[x]$?
  - What is the fundamental theorem of arithmetic in $\mathbb{F}[x]$?
  - What are the units in $\mathbb{F}[x]$?
  - What are the zerodivisors in $\mathbb{F}[x]$?
  - What is the GCD of two elements in $\mathbb{F}[x]$?
  - What is an irreducible element in $\mathbb{F}[x]$?
  - What special property do irreducible element for dividing other numbers?
  - Is the GCD of two polynomials in $\mathbb{F}[x]$ a linear combination? How do you find it as such?
  - What are the ideals in $\mathbb{F}[x]$?
- $R[x]$ for a general ring R
  - When is $R[x]$ a domain, if R is commutative?
  - If R is a domain, what are the units in $R[x]$?
  - Is every ideal of $R[x]$ generated by one element, in general?

  - Is there a division algorithm for $R[x]$ in general?
  - How is $\deg(fg)$ related to $\deg(f)$ and $\deg(g)$?
  - What if R is a domain?
- Quotient rings
  - What is a quotient ring?
  - What are the elements in R/I?
  - What are the operations in R/I?
  - What does the first isomorphism theorem say?
- $\mathbb{F}[x]/(f)$, for $\mathbb{F}$ a field and $f \in \mathbb{F}[x]$
  - What is $\mathbb{F}[x]/(f)$? What are its elements?
  - What are the units in $\mathbb{F}[x]/(f)$?
  - What are the zerodivisors in $\mathbb{F}[x]/(f)$?
  - When is $\mathbb{F}[x]/(f)$ a field? A domain?
  - For which f does $ax = b$ always have a solution in $\mathbb{F}[x]/(f)$, if $a \neq 0$?
  - How do you find inverses in $\mathbb{F}[x]/(f)$?
  - How do you solve $ax = b$ in $\mathbb{F}[x]/(f)$, when it does have a solution?
  - What are the ideals in $\mathbb{F}[x]/(f)$?