Math 412. Adventure sheet on Ring Basics

DEFINITION: A **ring**[1] is a nonempty set R with two binary operations, denoted "$+$" and "$\times$," such that

- $+$ and $\times$ are both associative,
- $+$ is commutative,
- $+$ has a identity, which we denote 0_R;
- Every element of R has an inverse for the operation $+$; the inverse of r is denoted $-r$.
- $\times$ has an identity, denoted 1_R;
- The two operations are related by the *distributive properties*: $a \times (b+c) = a \times b + a \times c$ and $(a+b) \times c = a \times c + b \times c$ for all $a, b, c \in R$.

If we want to specify the operations of the ring, we may write $(R, +, \times)$.

A. EXAMPLES OF RINGS. Which of the following have a natural ring structure? Describe the addition, multiplication, and identity elements.

(1) The set of polynomials $\mathbb{R}[x]$ with coefficients in $\mathbb{R}$.
(2) The set $\mathcal{P}_d$ of polynomials in $\mathbb{R}[x]$ of degree at most d.
(3) The set $M_2(\mathbb{R})$ of 2×2 matrices with coefficients in $\mathbb{R}$.
(4) The *Gaussian integers* $\mathbb{Z}[i] = \{a + bi \,|\, a, b \in \mathbb{Z}\}$.
(5) The set $\{even, odd\}$.
(6) The set $\mathbb{Z}_N$ for any positive integer N.
(7) The set of all 2×3 matrices with coefficients in $\mathbb{Z}$.
(8) The set of all 2×2 matrices with coefficients in $\mathbb{R}$ and non-zero determinant.
(9) The set $\mathcal{C}(\mathbb{R})$ of continuous functions from $\mathbb{R}$ to itself.
(10) The set of increasing functions from $\mathbb{R}$ to iself.

B. EASY PROOFS: Let $\Box$ be an operation on a set T.

(1) Prove that $\Box$ has at most one identity. Can a ring R have more than one 0_R? What about more than one 1_R? Explain.
(2) Suppose $\Box$ is associative. Prove that the $\Box$-inverse of any element $x \in T$, if it exists, is unique. Can elements of rings have more than one additive/multiplicative inverses?
(3) Prove that in any ring R, $0_R \times x = 0_R$ for all $x \in R$.[2]
(4) TRUE OR FALSE: In any ring R, $(a+b)^2 = a^2 + 2ab + b^2$ for all $a, b \in R$.

C. PRODUCT RINGS. Let R and S be rings.[3] Let $R \times S$ be the set of ordered pairs $\{(r, s) \mid r \in R, s \in S\}$.

(1) Define an operation called $+$ on $R \times S$ by $(r_1, s_1) + (r_2, s_2) = (r_1 + r_2, s_1 + s_2)$. The three different plus signs in the preceding sentence have three different meanings; explain.
(2) Is this operation on $R \times S$ is associative and commutative? Does it have an identity? Does every $(r, s) \in R \times S$ have an inverse under $+$?
(3) Define a natural multiplication on $R \times S$ so that, together with the addition defined in (1), the set $R \times S$ becomes a ring.
(4) How many elements are in the ring $\mathbb{Z}_N \times \mathbb{Z}_M$?

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[1] WARNING: Our definition requires that a ring have a multiplicative identity. The is different from the book's.

[2] Hint: Use the distributive law with lots of 0's.

[3] We always mean "ring with identity" when we say "ring;" this differs from the book's convention

(5) Make tables for the addition and multiplication in $\mathbb{Z}_2 \times \mathbb{Z}_2$. Identify the multiplicative and additive identity elements. Which elements have a multiplicative inverse?

D. SUBRINGS. A nonempty subset S of a ring R is a **subring** of R if S is a ring with the same operations $+, \times$ restricted to S.

(1) If $S \subseteq R$ is nonempty and R is a ring, explain why it suffices to show that
- $0_R, 1_R \in S$,
- S is *closed under addition*: $s_1, s_2 \in S \Rightarrow s_1 + s_2 \in S$,
- S is *closed under additive inverses*: $s_1 \in S \Rightarrow -s_1 \in S$, and
- S is *closed under multipication*: $s_1, s_2 \in S \Rightarrow s_1 s_2 \in S$,

to show that S is a subring of R.

(2) Find subrings of each of $\mathbb{Q}$, $\mathbb{R}[x]$, and $\mathrm{Fun}(\mathbb{R}, \mathbb{R})$.[4]

(3) Every ring always contains at least two "trivial" subrings. Explain.

E. ISOMORPHISM. Two rings R and S are **isomorphic** if they are "the same after relabelling." More precisely, two rings are **isomorphic** if there is a **bijection** (called an **isomorphism**) between them that preserves addition and multiplication. We write $R \cong S$.

(1) Write out the + and $\times$ tables for $\mathbb{Z}_4$. There is a way to relabel the elements of $\mathbb{Z}_4$ with the names $a, b, c,$ and d, so that they each become one of operations ♣, ♢, ♡, and ♠ from below. Explain.

(2) Now use the card suits to put a ring structure on the set $S = \{a, b, c, d\}$. Find the additive and multiplicative identities. Explain why S is isomorphic to $\mathbb{Z}_4$.

(3) Prove or disprove: $\mathbb{Z}_2 \times \mathbb{Z}_2$ is isomorphic to $\mathbb{Z}_4$.

(4) Prove or disprove: *Isomorphism is an equivalence relation on the set of all rings.* This means that it is reflexive ($R \cong S$), symmetric ($R \cong S$ implies $S \cong R$) and transitive ($R \cong S$ and $S \cong T$ implies $S \cong T$).

♣	a	b	c	d
a	a	b	c	d
b	b	b	c	d
c	c	c	c	d
d	d	d	d	d

♢	a	b	c	d
a	a	d	c	b
b	b	a	d	c
c	c	b	a	d
d	d	c	b	a

♡	a	b	c	d
a	a	a	a	a
b	a	b	c	d
c	a	c	a	c
d	a	d	c	b

♠	a	b	c	d
a	a	b	c	d
b	b	c	d	a
c	c	d	a	b
d	d	a	b	c

[4]The set of all functions from $\mathbb{R}$ to itself with the "pointwise" operations $(f + g)(x) = f(x) + g(x)$ and $(fg)(x) = f(x)g(x)$.