Math 412. Adventure sheet on Ring Homomorphisms

DEFINITION: A **ring homomorphism** is a mapping $R \xrightarrow{\phi} S$ between two rings (with identity) that satisfies:

(1) $\phi(x + y) = \phi(x) + \phi(y)$ for all $x, y \in R$.
(2) $\phi(x \cdot y) = \phi(x) \cdot \phi(y)$ for all $x, y \in R$.
(3) $\phi(1) = 1$.

DEFINITION: A **ring isomorphism** is a bijective ring homomorphism. We say that two rings R and S are **isomorphic** if there is an isomorphism $R \to S$ between them.

You should think of an isomorphism as a renaming: isomorphic rings are "the same ring" with the elements named differently.

DEFINITION: The **kernel** of a ring homomorphism $R \xrightarrow{\phi} S$ is the set of elements in the source that map to the ZERO of the target; that is,

$$\ker \phi = \{r \in R \mid \phi(r) = 0_S\}.$$

THEOREM: A ring homomorphism $R \to S$ is injective if and only if its kernel is zero.

A. EXAMPLES OF HOMOMORPHISMS: Which of the following mappings between rings is a **homomorphism**? Which are **isomorphisms**?

(1) The inclusion mapping $\mathbb{Z} \hookrightarrow \mathbb{Q}$ sending each integer n to the rational number $\frac{n}{1}$.
(2) The doubling map $\mathbb{Z} \to \mathbb{Z}$ sending $n \mapsto 2n$.
(3) The **residue map** $\mathbb{Z} \to \mathbb{Z}_n$ sending each integer a to its congruence class $[a]_n$.
(4) The "evaluation at 0" map $\mathbb{R}[x] \to \mathbb{R}$ sending $f(x) \mapsto f(0)$.
(5) The differentiation map $\mathbb{R}[x] \to \mathbb{R}[x]$ sending $f \mapsto \frac{df}{dx}$.
(6) The map $\mathbb{R} \to M_2(\mathbb{R})$ sending $\lambda \mapsto \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$.
(7) The map $\mathbb{R} \to M_2(\mathbb{R})$ sending $\lambda \mapsto \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix}$.
(8) The map $M_2(\mathbb{Z}) \to \mathbb{R}$ sending each 2×2 matrix to its determinant.

B. BASIC PROOFS: Let $\phi : S \to T$ be a homomorphism of rings.

(1) Show that $\phi(0_S) = 0_T$.
(2) Show that for all $x \in S$, $-\phi(x) = \phi(-x)$. That is: a ring homomorphism "respects additive inverses".
(3) Show that if $u \in S$ is a unit, then also $\phi(u) \in T$ is a unit.

C. KERNEL OF RING HOMOMORPHISMS: Let $\phi : R \to S$ be a ring homomorphism

(1) Prove that $\ker \phi$ is nonempty.
(2) Compute the kernel of the canonical homomorphism: $\mathbb{Z} \to \mathbb{Z}_n$ sending $a \mapsto [a]_n$.
(3) Compute the kernel of the homomorphism $\mathbb{Z} \to \mathbb{Z}_7 \times \mathbb{Z}_{11}$ sending $n \mapsto ([n]_7, [n]_{11})$.
(4) For arbitrary rings R, S, compute the kernel of the projection homomorphism $R \times S \to R$ sending $(r, s) \mapsto r$. Write your answer in set-builder notation.

D. Prove the THEOREM: *A ring homomorphism $\phi : R \to S$ is injective if and only if its kernel is ZERO.*

E. ISOMORPHISM. Suppose that $R \xrightarrow{\phi} S$ is a ring isomorphism. True or False.

(1) ϕ induces a bijection between units.
(2) ϕ induces a bijection between zerodivisors.
(3) R is a field if and only if S is a field.
(4) R is an (integral) domain if and only if S is an (integral) domain.

F. ISOMORPHISM. Consider the set $S = \{a, b, c, d\}$ and the associative binary operations $\heartsuit$ and $\spadesuit$ listed below. You observed earlier that $(S, \spadesuit, \heartsuit)$ is a ring.

♡	a	b	c	d
a	a	a	a	a
b	a	b	c	d
c	a	c	a	c
d	a	d	c	b

♠	a	b	c	d
a	a	b	c	d
b	b	c	d	a
c	c	d	a	b
d	d	a	b	c

⊕	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	a	b
d	d	c	b	a

⊗	a	b	c	d
a	a	a	a	a
b	a	b	c	d
c	a	c	c	a
d	a	d	a	d

(1) Discuss and recall some of the features of the ring $(S, \spadesuit, \heartsuit)$. To what more familiar ring is it isomorphic?
(2) The operations $\oplus$ and $\otimes$ (whose tables are listed above) define a *different* ring structure on S. What are the zero and one? Is the ring $(S, \oplus, \otimes)$ isomorphic to $(S, \spadesuit, \heartsuit)$? Explain.
(3) Find an **explicit** isomorphism $\mathbb{Z}_2 \times \mathbb{Z}_2 \to (S, \oplus, \otimes)$.
(4) Can you find a *different* isomorphism $\mathbb{Z}_2 \times \mathbb{Z}_2 \to (S, \oplus, \otimes)$?
(5) Can there be more than one isomorphism $\mathbb{Z}_4 \to (S, \spadesuit, \heartsuit)$?

G. CAUTIONARY EXAMPLES: Let R and S be arbitrary rings.

(1) Is the map $R \to R \times S$ sending $a \mapsto (a, 1)$ a ring homomorphism? Explain.
(2) Is the map $R \to R \times S$ sending $a \mapsto (a, 0)$ a ring homomorphism? Explain.
(3) Find a natural homomorphism $R \to R \times R$. Can it be an isomorphism?

H. CANONICAL RING HOMOMORPHISMS: Let R be any ring[1]. Prove that there exists a **unique** ring homomorphism $\mathbb{Z} \to R$.

[1]with identity of course