Name:

Math 412 Winter 2019 Midterm Exam

Time: 100 mins.

1. Answer each question in the space provided. If you require more space, you may use the blank page at the end of this exam, but you must clearly indicate in the provided answer space that you have done so.
2. You may use any results proved in class, on the homework, or in the textbook, except for the specific question being asked. You should clearly state any facts you are using.
3. Remember to show all your work.
4. No calculators, notes, or other outside assistance allowed.

Best of luck!

Problem	Score
1	
2	
3	
4	
5	
6	
7	
8	
Total	

Problem 1 (20 points)**.** Write complete, precise definitions for, or precise mathematical characterizations of, each of the following italicized terms. Be sure to include any quantifiers as needed.

a) An *ideal* I in a ring R.

b) A *field* $\mathbb{F}$.

c) The *additive inverse* y of an element x in a ring R.

d) The *congruence class* of a given integer n modulo 17.

Problem 2 (12 points)**.** For each of the questions below, give an example with the required properties. No explanations required.

a) A domain that is not a field.

b) A surjective ring homomorphism that is not injective.

c) A nonzero nonunit (element that is not a unit) in $\mathbb{Z}_{4699}$.

Note: $4699 = 127 \cdot 37$ is a prime factorization.

d) A finite subring of an infinite ring.

Problem 3 (20 points)**.** For each of the questions below, indicate clearly whether the statement is *true* or *false*, and give a short justification.

a) There are 11 distinct principal ideals in the ring $\mathbb{Z}_{11}$.

b) Every subring of a commutative ring is commutative.

c) If $\mathbb{F}$ is a field, then the polynomial ring $\mathbb{F}[x]$ is a field.

d) If R is a ring in which $0_R \neq 1_R$, and $\varphi : M_2(\mathbb{R}) \to R$ is a homomorphism, then φ must be injective.

Problem 4 (10 points)**.** Consider the following operation table for an associative operation $*$ on the set $S = \{a, b, c, d, e, f\}$: here the entry in row x and column y corresponds to the value of $x * y$. Use the table to answer the following questions.

$*$	a	b	c	d	e	f
a	a	b	c	d	e	f
b	b	c	a	e	f	d
c	c	a	b	f	d	e
d	d	f	e	a	c	b
e	e	d	f	b	a	c
f	f	e	d	c	b	a

a) Does $*$ have an identity, and if so, what is it?

b) Is the operation $*$ commutative?

c) Can the operation $*$ be the multiplication for some ring? Justify your answer.

Problem 5 (12 points)**.** For each of the following elements of various rings, find a multiplicative inverse, or else explain why no multiplicative inverse exists.

a) $[26]_{57} \in \mathbb{Z}_{57}$.

b) $2x + 1 \in \mathbb{Z}[x]$.

c) $\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \in M_2(\mathbb{Z}_3)$.

Problem 6 (10 points)**.** Consider $\mathcal{L} = \left\{ \begin{bmatrix} a & 0 \\ 2b & c \end{bmatrix} : a, b, c \in \mathbb{Z} \right\} \subseteq M_2(\mathbb{Z})$.

a) Show that $\mathcal{L}$ is a subring of $M_2(\mathbb{Z})$.

b) Let I be the ideal of $\mathcal{L}$ of matrices with zeroes on the diagonal:

$$I = \left\{ \begin{bmatrix} 0 & 0 \\ 2b & 0 \end{bmatrix} : b \in \mathbb{Z} \right\} \subseteq \mathcal{L}.$$

Consider the matrix

$$A = \begin{bmatrix} -1 & 0 \\ 2 & -1 \end{bmatrix}.$$

Show that $(A + I)^2 = 1 + I$ in $\mathcal{L}/I$.

Problem 7 (6 points)**.** Let $f(x), h(x), j(x) \in \mathbb{R}[x]$. Prove that[1] if $f(x)g(x) \equiv h(x) \mod j(x)$ has a solution $g(x)$, then $\gcd(f(x), j(x)) \mid h(x)$.

[1] Recall that $a(x) \equiv b(x) \mod j(x)$ here means $a(x)$ is congruent to $b(x)$ modulo the ideal $I = (j(x))$ generated by $j(x)$.

Problem 8 (10 points).

a) How many different ring homomorphisms $\mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow \mathbb{Z}_4$ are there?

b) How many different ring homomorphisms $\mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2$ are there?