Math 412. Adventure sheet on operations and the definition of rings

DEFINITION: An **operation** on a set S is a function from $S \times S$ to S.

For example, addition and subtraction are operations on set the of integers (or on the set of real numbers). We might write $\star$ for an operation, and write $x \star y$ to indicate the result of applying an operation to (x, y), just as we would with the symbols $+$, $-$, etc.

A **ring** is a set with two operations, which we usually call addition and multiplication, that behave in similar ways to addition and multiplication of numbers. To make this precise, we specify some special abstract properties of operations.

- *Commutativity.* An operation $\star$ is **commutative** if $x \star y = y \star x$ for any $x, y \in S$.
  - Find an example of an operation on the set of 2×2 matrices that *is* commutative, and an example of an operation on the same set that *is not* commutative. Can you think of more than one of each?

- *Associativity.* By definition, operations only take two inputs. If we wanted to operate on three things, we would have to choose two to pair first, then throw in the third. An operation $\star$ is **associative** if we get the same result with either grouping: $(x \star y) \star z = x \star (y \star z)$ for any x, y, z in S.
  - Find an example of an operation on $\mathbb{Z}$ that *is* associative, and an example of an operation on $\mathbb{Z}$ that *is not* associative.
- Let S be the set of functions $X \to X$ for some other set X. Prove that the operation on S "composition of functions" is associative.
- Can you find an example of an operation on a set that *is* associative but *is not* commutative? What about the other way around? [1]

- *Identity.* An element $e \in S$ is an **identity** for $\star$ if $e \star x = x \star e = x$ for all $x \in S$.
  - Which of the following operations have an identity? If so, what is it:
    a) addition on the set $\mathbb{R}[x]$ of real polynomials
    b) subtraction on the set $\mathbb{R}[x]$ of real polynomials
    c) multiplication of 2×2 matrices
    d) division of positive real numbers
    e) composition of functions
    f) averaging two rational numbers
    g) maximum of two rational numbers
  - Prove that any operation has at most one identity.

- *Inverses.* If $\star$ is an operation with an identity e, then an **inverse** for an element x is another element y such that $x \star y = y \star x = e$.
  - For each of the operations above that has an identity, does it have an inverse? How do you find inverses for your operation?

[1] Hint: Maybe something on the list of operations under "identity" works.

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g) maximum of two rational numbers

- Prove that any operation has at most one identity.

- *Inverses.* If $\star$ is an operation with an identity e, then an **inverse** for an element x is another element y such that $x \star y = y \star x = e$.
  - For each of the operations above that has an identity, does it have an inverse? How do you find inverses for your operation?