Name: Solutions

Problem 1 (3 points)**.** Define normal subgroup.

A subgroup N of G is normal if for all $g \in N$, $gN = Ng$ (as sets).

Problem 2 (4 points)**.** Prove that the kernel of a group homomorphism is a normal subgroup.

Let $f: G \to H$ be a group homomorphism.

- $e_G \in \ker f$, since $f(e_G) = e_H$.
- if $g, h \in \ker f$ $\quad f(gh) = f(g)f(h) = e_H e_H = e_H$ so $gh \in \ker f$
- if $g \in \ker f$, $f(g^{-1}) = f(g)^{-1} = e_H^{-1} = e_H$, so $g^{-1} \in \ker f$.

Finally, let $g \in G$, $h \in \ker H$. then $ghg^{-1} \in \ker f$, since

$$f(ghg^{-1}) = f(g)\, f(\underbrace{h^{-1}}_{\in \ker f})\, f(g^{-1}) = f(g) f(g^{-1}) = f(gg^{-1}) = e_H.$$

this shows that $\ker f$ is a normal subgroup.

Problem 3 (3 points)**.** True or false: a group of order 15 can act on a set with 7 elements in such a way that there are exactly 2 orbits.

False. Suppose that a group G of order 15 does act on a set X with 7 elements, and there are exactly two orbits, $O_1 = O(x_1) \neq O_2 = O(x_2)$. By the Orbit-Stabilizer theorem, $|O_1|$, $|O_2|$ must both divide $|G| = 15$. On the other hand, X has 7 elements, so $|O_1|$ and $|O_2|$ can only be 1, 3 or 5.

Moreover, $X = O_1 \underset{\text{disjoint union}}{\uplus} O_2$, so $|O_1| + |O_2| = 7$. But this is impossible!

$1+3=4 \qquad 1+1=2$

$1+5=6 \qquad 3+3=6$

$3+5=8 \qquad 5+5=10$