Name: Solutions

Problem 1 (3 points)**.** Let G be a group and N be a subgroup of G. When does the set of cosets G/N have a natural group structure?

When N is a normal subgroup.
(that's the only way the "obvious" product rule is well-defined)

Problem 2 (4 points)**.** Show that $\mathbb{Z} \times \mathbb{Z}/\langle(1,1)\rangle \cong \mathbb{Z}$.

the map $\varphi : \mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{Z}$, $(a,b) \longmapsto a-b$ is a group homomorphism:

$$\varphi(a,b) + \varphi(c,d) = (a-b) + (c-d)$$
$$= (a+c) - (b+d) = \varphi((a+c, b+d))$$

Moreover, φ is surjective: given any $n \in \mathbb{Z}$, $n = \varphi(n,0)$.
Finally, $(a,b) \in \ker\varphi \Leftrightarrow a-b=0 \Leftrightarrow a=b \Leftrightarrow (a,b) \in \langle(1,1)\rangle$.
By the First Isomorphism Theorem, $\mathbb{Z}\times\mathbb{Z}/\langle(1,1)\rangle \cong \mathbb{Z}$.

Problem 3 (3 points)**.** True or false: any two elements in the same conjugacy class have the same order.

True. If $h, h' \in G$ are in the same conjugacy class, that means there exists $g \in G$ such that $h' = ghg^{-1}$.
Given any $n > 1$,

$$(ghg^{-1})^n = \underbrace{(ghg^{-1})(ghg^{-1}) \cdots (ghg^{-1})}_{n \text{ times}} = gh^n g^{-1}$$

then $gh^ng^{-1} = e \Leftrightarrow h^n = g^{-1}g = e$.

So $|h| = |h'|$.