Name: Solutions

Problem 1 (4 points). For each pair $(G, \cdot)$, where $\cdot$ is an operation on the set G, circle all the ones that are groups.

a) $(\mathbb{Z}, +)$.

b) $(\mathbb{Z}, \times)$. no inverses

c) $(\mathbb{N}, +)$. no inverses

d) $(\mathbb{R}_{>0}, \times)$.

e) $(\mathbb{R}_{\geqslant 0}, +)$. no inverses

f) $(\mathbb{Z}_{27}, +)$.

Problem 2 (3 points). State the first isomorphism theorem for rings.

Let $R \xrightarrow{f} S$ be a surjective ring homomorphism.
then $R/\ker f \cong S$.

Problem 3 (3 points). Consider the ring $R = \mathbb{Z}[x]$ and the ideal

$$I = \{p(x) \in R \text{ such that } 3 \mid p(0)\}.$$

Use the first isomorphism theorem for rings to prove that $R/I \cong \mathbb{Z}_3$.

Consider the map $f: R \longrightarrow \mathbb{Z}_3$ given by

$$f(p(x)) = [p(0)]_3$$

this is a ring homomorphism: $f(1) = [1]_3$, and for all $p(x), q(x) \in R$,

$$f(p(x) + q(x)) = [p(0) + q(0)]_3 = [p(0)]_3 + [q(0)]_3 = f(p(x)) + f(q(x))$$

$$f(p(x)q(x)) = [p(0)q(0)]_3 = [p(0)]_3 [q(0)]_3 = f(p(x)) f(q(x))$$

Moreover, f is surjective: given any element $a \in \mathbb{Z}_3$ and an integer n with $[n]_3 = a$, $f(\underbrace{n}_{\text{constant polynomial}}) = [n]_3 = a$.

Notice that $p(x) \in \ker f \iff 3 \mid p(x)$, so $\ker f = I$.
Therefore, by the First Isomorphism Theorem, $R/I \cong \mathbb{Z}_3$