Math 871 Midterm 2 solutions

Problem 0. The problems on this page require **no proof or explanation**.

a) Define continuous function.

Solution. A function $f\colon X \longrightarrow Y$ between topological spaces is continuous if for all open sets U in Y, $f^{-1}(U)$ is open in X.

b) Define homeomorphism.

Solution. A continuous bijection between topological space with continuous inverse.

c) Give an explicit basis for the product topology on $\mathbb{R}^\omega$.

Solution. The set of all subsets of the form

$$(a_1, b_1) \times \cdots \times (a_n, b_n) \times \prod_{i \geqslant n+1} \mathbb{R}$$

where $n \geqslant 1$ and $a_i < b_i$ are real numbers.

d) Give an example of a topological space that is Hausdorff.

Solution. $\mathbb{R}$ with the standard topology.

Problem 1.

a) Show that every metric space is Hausdorff.

Solution. Let d be a metric on X that generates the topology. For $x, y \in X$ with $x \neq y$, set $\varepsilon = \frac{1}{2}d(x, y)$. Then $B^d_\varepsilon(x)$ and $B^d_\varepsilon(y)$ are open neighborhoods of a and b that do not meet: if $z \in B^d_\varepsilon(x)$, then by the Triangle Inequality

$$d(x, z) + d(z, y) \geqslant d(x, y) = \varepsilon \implies d(z, y) \geqslant \varepsilon - d(x, z) = \varepsilon - \frac{\varepsilon}{2} = \frac{\varepsilon}{2}$$

and thus $z \notin B^d_\varepsilon(y)$.

b) Endow $\mathbb{R}$ with the standard topology. Show that $\mathbb{R}$ with the finite complement topology is not a metrizable space.

Solution. It suffices to show that $\mathbb{R}$ in the finite complement topology is not Hausdorff. Fix any two nonempty open sets U and V. If $U \cap V = \emptyset$, then in particular $V \subseteq \mathbb{R} \setminus U$. Since U is a nonempty open set, $\mathbb{R} \setminus U$ is finite, so V is finite. But V is a nonempty open set, and thus its complement also finite. Since $\mathbb{R}$ is infinite, this is impossible. We conclude that any two nonempty open sets intersect nontrivially. In particular, given $x \neq y$ and open sets $U \ni x$ and $V \ni y$, $U \cap V \neq \emptyset$. We conclude that this space is not Hausdorff, and thus by (a) it is not metrizable.

Problem 2. Let X and Y be topological spaces and let $f\colon X \longrightarrow Y$ be any function.

a) Show that if f is continuous, then f sends convergent sequences to convergent sequences, meaning that for every sequence (x_n) in X converging to x, the sequence $(f(x_n))$ converges to $f(x)$.

Solution. Let (x_n) be a sequence of points in X that converges to some point $x \in X$. Let U be any open neighborhood of $f(x)$ in Y. Since f is continuous and $f(x) \in U$, $f^{-1}(U)$ is an open neighborhood of x in X. Since (x_n) converges to x, there exists some N such that $x_n \in f^{-1}(U)$ for all $n \geqslant N$. Therefore, $f(x_n) \in U$, for all $n \geqslant N$. This proves that $(f(x_n))$ converges to $f(x)$.

b) We say X is **Fréchet-Urysohn** if for every subset $S \subseteq X$ and every $s \in \overline{S}$, there exists a sequence (s_n) of points in S converging to s. Show that if X is Fréchet-Urysohn and f sends convergent sequences to convergent sequences, then f is continuous.

Solution. Given any closed set C in Y, we will show its preimage $f^{-1}(C)$ is closed. If $f^{-1}(C) = \emptyset$, then it is closed. Suppose $f^{-1}(C) \neq \emptyset$, and let $x \in \overline{f^{-1}(C)}$.

Since X is Fréchet-Urysohn, there exists a sequence of points $(x_n) \in f^{-1}(C)$ converging to x. Therefore, $(f(x_n))$ is a sequence in Y converging to $f(x)$. On the other hand, $(f(x_n))$ is a sequence of points in C. If $f(x) \notin C$, then we would be able to find an open neighborhood U of $f(x)$ such that $U \cap C = \emptyset$, and in particular U would contain none of the points in $(f(x_n))$. Therefore, $f(x) \in C$, so $x \in f^{-1}(C)$. We conclude that $f^{-1}(C)$ is closed and thus f is continuous.

Problem 3. Choose **one** of the following two problems:

A. Endow $\mathbb{R}$ with the standard topology. Show that the product and uniform topologies on $\mathbb{R}^\omega$ do not coincide.

Solution. In $\mathbb{R}^\omega$ with the uniform metric $\overline{\rho}$, consider the point $z = (0)$, and the open ball $B_{\frac{1}{2}}(z)$. Any basic open neighborhood of z in the product topology, of the form

$$U = \prod_{i=1}^{\infty} U_i$$

in the product topology, with $U_i = (a_i, b_i)$ for finitely many i, say $i \leqslant N$, and $U_i = \mathbb{R}$ for all other i, contains the point

$$y = (y_n)_{n \geqslant 1} \quad \text{with} \quad y_n = \begin{cases} 0 & n \leqslant N \\ 1 & n > N. \end{cases}$$

However,

$$d(z, y) = 1 \implies y \notin B_{\frac{1}{2}}(z).$$

This shows that the product and uniform topologies on $\mathbb{R}^\omega$ are distinct.

B. Show that the arbitrary product of Hausdorff spaces is Hausdorff in the product topology.

Solution. Consider any two distinct elements $x = (x_\alpha)_{\alpha \in I}$ and $y = (y_\alpha)_{\alpha \in I}$ in the product. Then there exists $\beta \in I$ such that $x_\beta \neq y_\beta$.

Since X_β is Hausdorff, we can find neighborhoods A of x_β and B of y_β such that $A \cap B = \emptyset$. Set

$$U = \prod_{\alpha \in I} U_\alpha \text{ with } U_\alpha = \begin{cases} A & \text{if } \alpha = \beta \\ X_\alpha & \text{if } \alpha \neq \beta \end{cases} \quad \text{and} \quad V = \prod_{\alpha \in I} V_\alpha \text{ with } V_\alpha = \begin{cases} B & \text{if } \alpha = \beta \\ X_\alpha & \text{if } \alpha \neq \beta. \end{cases}$$

By definition of the product topology, U and V are open sets in the product. Moreover, if $(z_\alpha) \in U \cap V$, then in particular $z_\beta \in A \cap B = \emptyset$, so $U \cap V = \emptyset$.

Problem 4. Choose **one** of the following two problems.

A. Let $f \colon X \to Y$ and $g \colon X \to Y$ be a continuous function with Y a Hausdorff space. Show that $C = \{x \in X \mid f(x) = g(x)\}$ is a closed subset of X.

Solution. Let $x \notin C$, so that $f(x) \neq g(x)$. Since Y is Hausdorff, we can find open neighborhoods U of $f(x)$ and V of $g(x)$ such that $U \cap V = \emptyset$. Since f and g are continuous functions, $f^{-1}(U)$ and $g^{-1}(V)$ are open subsets of X. Moreover,

$$f(x) \in U \text{ and } g(x) \in V \implies x \in f^{-1}(U) \cap g^{-1}(V).$$

Finally, suppose that $z \in f^{-1}(U) \cap g^{-1}(V)$, so $f(z) \in U$ and $g(z) \in V$. Since $U \cap V = \emptyset$, we conclude that $f(z) \neq g(z)$, and thus $z \notin C$. We conclude that $f^{-1}(U) \cap g^{-1}(V)$ is an open neighborhood of x that intersects C trivially. Since this holds for all $x \notin C$, C must be closed.

Alternative solution. By the universal property of the product, $h \colon X \longrightarrow Y \times Y$ given by $h(x) = (f(x), g(x))$ is continuous (when Y is given the product topology). Moreover, since Y is Hausdorff, by a problem set we know that

$$\Delta_Y = \{(y, y) \mid y \in Y\} \subseteq Y \times Y$$

is a closed subset of $Y \times Y$. Behold:

$$h^{-1}(\Delta_Y) = C.$$

Since h is continuous and Δ_Y is closed, then C is a closed subset of X.

B. Let X and Y be topological spaces. A function $f\colon X \longrightarrow Y$ is locally constant if for all $x \in X$ there exists an open neighborhood U of x such that $f|_U$ is constant. Show that every locally constant function is continuous.

Solution. Suppose that f is locally constant , and let V be an open set in Y. If $f^{-1}(V)$ is empty, then it is open. If $f^{-1}(V)$ is nonempty, let $x \in f^{-1}(V)$. Since f is locally constant, there exists an open neighborhood U of x such that $f|_U$ is constant. For all $z \in U$, $f(z) = f(x) \in V$, and thus $z \in f^{-1}(V)$. We conclude that for all $x \in f^{-1}(V)$ there exists an open set U with

$$x \in U \subseteq f^{-1}(V)$$

and thus $f^{-1}(V)$ is open.

Alternative solution. Suppose that f is locally constant . For each $x \in X$, fix an open set U_x such that $f|_{U_x}$ is constant. Since constant functions are continuous, $f|_{U_x}$ is continuous. Moreover,

$$\bigcup_{x \in X} U_x = X.$$

By the Pasting Lemma, f is continuous.