Math 871 Topology I
Problem Set 10 solutions

1. Let A and B nonempty open subsets of a topological space Z. Prove that if both $A \cap B$ and $A \cup B$ are connected, then each of A and B is connected.

   Solution. Since A is open in Z, any open subset of A is open in Z (as the intersection of open sets is open). Assume, by contradiction, that A is disconnected, and let $U \subseteq A$ and $V \subseteq A$ be open sets in Z such that (U, V) is a separation for A.

   Since $A \cap B \subseteq A$ is connected, then $A \cap B \subseteq U$ or $A \cap B \subseteq V$. Assume, without loss of generality, that $A \cap B \subseteq U$, which implies that $(A \cap B) \cap V = \emptyset$. Since $V \subseteq A$, we conclude that $V \cap B = \emptyset$. Since U and V are also disjoint, we conclude that V and $U \cup B$ are disjoint.

   Since U and B are open in Z, then $U \cup B$ is open in Z. Thus V and $U \cup B$ are nonempty disjoint open subsets of Z and

   $$A \cup B = (V \cup U) \cup B = V \cup (U \cup B).$$

   We conclude that $A \cup B$ is disconnected.

   Important note: The statement is false if we do not assume that A and B are both open. Here is an example: take $Z = \mathbb{R}$, $A = [0, 1]$, and $B = Z \setminus [0, 1]$. Then $A \cap B = \emptyset$ and $A \cup B = \mathbb{R}$ are both connected, and yet B is disconnected.

2. Let X be a connected space and endow $\mathbb{R}$ with the standard topology and $[0, 1] \subseteq \mathbb{R}$ with the subspace topology. Let $f, g \colon X \longrightarrow [0, 1]$ be continuous functions and assume that f is surjective. Show that there exists $x \in X$ with $f(x) = g(x)$.

   Solution. Composing f and g with the inclusion of $[0, 1]$ into $\mathbb{R}$, which is continuous, we get continuous functions $X \longrightarrow \mathbb{R}$. Consider the function $h \colon X \longrightarrow \mathbb{R}$ given by $h(x) = f(x) - g(x)$, which is continuous by a previous problem set.

   Since f is surjective, there exist $a, b \in X$ such that $f(a) = 0$ and $f(b) = 1$, and thus

   $$h(a) = -g(a) \leqslant 0 \qquad \text{and} \qquad h(b) = 1 - g(b) \geqslant 1 - 1 = 0.$$

   Since X is connected, we can apply the Intermediate Value Theorem, which gives us $x \in X$ such that

   $$h(x) = 0 \iff f(x) = g(x).$$

3. Endow $\mathbb{R}$ with the standard topology. Show that for all continuous maps $f \colon S^1 \longrightarrow \mathbb{R}$, there exists $x \in S^1$ such that $f(x) = f(-x)$.

   Solution. Consider the function $g \colon S^1 \longrightarrow \mathbb{R}$ given by $g(x) = f(x) - f(-x)$. We want to show that $g(x) = 0$ for some $x \in S^1$.

   First, note that g is continuous: by a previous problem set, the difference of continuous functions is continuous, and the function $S^1 \longrightarrow \mathbb{R}$ given by $x \mapsto f(-x)$ is the

composition of two continuous functions (again using the same previous problem set to say that the additive inverse is a continuous function).

If g is the constant function equal to zero, then we are done, so assume that $g(y) \neq 0$ for some $y \in \mathbb{R}$. Without loss of generality, suppose that $g(y) > 0$. Then

$$g(-y) = f(-y) - f(y) = -(f(y) - f(-y)) = -g(y) < 0.$$

By the Intermediate Value Theorem, there exists $z \in S^1$ such that $g(z) = 0$, as desired.

4. Let X and Y be topological spaces and $f\colon X \longrightarrow Y$ a continuous function. Define an equivalence relation on X by $x_1 \sim x_2$ if and only if $f(x_1) = f(x_2)$. Show that if X is connected and Y is Hausdorff, then the quotient space $X/\sim$ with the quotient topology is both connected and Hausdorff.

Solution. Let $p\colon X \longrightarrow X/\sim$ be the canonical map, which is surjective by construction and continuous by definition of the quotient topology. Since X is connected, then $p(X) = X/\sim$ is connected, as the image of a connected space by a continuous function is continuous.

By definition of the equivalence relation, f is constant on all fibers, and thus by the universal property of the quotient, f induces the continuous function $\overline{f}\colon X/\sim \longrightarrow Y$ given by $\overline{f}([x]) = f(x)$. Moreover, $\overline{f}$ is injective, since

$$\overline{f}([x_1]) = \overline{f}([x_2]) \iff f(x_1) = f(x_2) \iff x_1 \sim x_2 \iff [x_1] = [x_2].$$

Now suppose that $[x_1] \neq [x_2]$ are two distinct elements in $X/\sim$. Then $\overline{f}([x_1]) \neq \overline{f}([x_2])$, and since Y is Hausdorff, there exist open sets U and V in Y such that $\overline{f}([x_1]) \in U$, $\overline{f}([x_2]) \in V$, and $U \cap V = \emptyset$. Since $\overline{f}$ is continuous, $\overline{f}^{-1}(U)$ and $\overline{f}^{-1}(V)$ are open sets in X, with $[x_1] \in \overline{f}^{-1}(U)$ and $[x_2] \in \overline{f}^{-1}(V)$ by construction. Finally, by elementary properties of preimages we have

$$\overline{f}^{-1}(U) \cap \overline{f}^{-1}(V) = \overline{f}^{-1}(U \cap V) = \overline{f}^{-1}(\emptyset) = \emptyset.$$

Alternative solution Let $p\colon X \longrightarrow X/\sim$ be the canonical map, which is surjective by construction and continuous by definition of the quotient topology. Since X is connected, then $p(X) = X/\sim$ is connected, as the image of a connected space by a continuous function is continuous.

Now suppose that $[x_1] \neq [x_2]$ are two distinct elements in $X/\sim$, so that $x_1 \nsim x_2$ in X. Then by definition of the equivalence relation, $f(x_1) \neq f(x_2)$, and since Y is Hausdorff, there exist open sets U and V in Y such that $f(x_1) \in U$, $f(x_2) \in V$, and $U \cap V = \emptyset$. Since f is continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are open sets in X. Moreover, $x_1 \in f^{-1}(U)$ and $x_2 \in f^{-1}(V)$. On the other hand, we claim that $f^{-1}(U)$ and $f^{-1}(V)$ are saturated with respect to p. To see that, suppose that $y \in f^{-1}(U)$ and $z \sim y$. Then $f(z) = f(y) \in U$, so $z \in f^{-1}(U)$. Therefore, $A = p(f^{-1}(U))$ is an open set in $X/\sim$. Similarly, $B = p(f^{-1}(V))$ is an open set in $X/\sim$.

Now we have $[x_1] = p(x_1) \in A$ since $x_1 \in f^{-1}(U)$ and analogously $[x_2] \in B$. Finally, we claim that $A \cap B = \emptyset$. Suppose $[z] \in A \cap B$. Since $f^{-1}(U)$ and $f^{-1}(V)$ are saturated with respect to p, this implies that any representative $z \in X$ of $[z]$ has $z \in f^{-1}(U) \cap f^{-1}(V)$. However, $U \cap V = \emptyset$ and by elementary properties of preimages

$$f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V) = f^{-1}(\emptyset) = \emptyset.$$

We conclude that for any distinct elements $[x_1], [x_2] \in X/\sim$, there exist open sets A and B with $[x_1] \in A$ and $[x_2] \in B$.