Math 871 Topology I
Problem Set 1

1. Let X be a set. Set
$$\mathcal{T} = \{U \subseteq X \mid X \setminus U \text{ is infinite}\} \cup \{\emptyset, X\}.$$
Show that $\mathcal{T}$ is a topology on X if and only if X is finite.

   Solution 1. Assume X is finite. Then $\mathcal{T} = \{\emptyset, X\}$ is the discrete topology.

   Assume X is infinite. For each $x \in X$, $X \setminus \{x\}$ is also infinite and thus $\{x\}$ is in $\mathcal{T}$. Fix $y \in X$. Then
$$\bigcup_{\substack{x \in X \\ x \neq y}} \{x\} = X \setminus \{y\},$$
which is not in $\mathcal{T}$, since $X \setminus (X \setminus \{y\}) = \{y\}$ is finite. Therefore, $\mathcal{T}$ is not closed for arbitrary unions, and thus $\mathcal{T}$ is not a topology.

2. Prove or disprove: is
$$\mathcal{T} = \{(-\infty, a] \mid a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$$
a topology on $\mathbb{R}$?

   Solution 2. Note that for all $n \geqslant 1$, the sets
$$U_n = \left(-\infty, -\frac{1}{n}\right]$$
are in $\mathcal{T}$, but
$$\bigcup_{n \geqslant 1} U_n = (-\infty, 0) \notin \mathcal{T}.$$
Thus this is not a topology, as it is not closed for unions.

3. Let $\{\mathcal{T}_\alpha\}_\alpha$ be a collection of topologies on a set X. For each of the following options, either show that it is always a topology or given an example where it is not, with proof.

   (a) Prove or disprove: is $\bigcap\limits_\alpha \mathcal{T}_\alpha$ always a topology on X?

   Solution 3. We claim that $\bigcap\limits_\alpha \mathcal{T}_\alpha$ is a topology on X.

   Suppose $\{\mathcal{T}_\alpha\}$ is a family of topologies on X. Since each $\mathcal{T}_\alpha$ is a topology, we have that $\emptyset \in \mathcal{T}_\alpha$ for all indices α, which means that $\emptyset \in \bigcap \mathcal{T}_\alpha$. Similarly, $X \in \mathcal{T}_\alpha$ for all α, and thus $X \in \bigcap \mathcal{T}_\alpha$.

   Consider a collection $\{U_\beta\}$ of elements of $\bigcap \mathcal{T}_\alpha$. Then for every α, we have that each $U_\beta \in \mathcal{T}_\alpha$, and since $\mathcal{T}_\alpha$ is a topology, it is also the case that $\bigcup U_\beta \in \mathcal{T}_\alpha$. As this is true for every α, it follows that $\bigcup U_\beta \in \bigcap \mathcal{T}_\alpha$, and $\mathcal{T}_\alpha$ satisfies the second property required to be a topology.

Finally, suppose that $U, V \in \bigcap \mathcal{T}_\alpha$. For any α, we again use the fact that $\mathcal{T}_\alpha$ is a topology to conclude that the intersection $U \cap V$ is again an element of $\mathcal{T}_\alpha$. Since this is true for every α, we have $U \cap V \in \bigcap \mathcal{T}_\alpha$. We conclude that $\bigcap T_\alpha$ is a topology on X.

(b) Prove or disprove: is $\bigcup_\alpha \mathcal{T}_\alpha$ always a topology on X?

Solution 4. We claim that $\bigcup \mathcal{T}_\alpha$ need not necessarily be a topology on X.

Here is a counterexample: consider the set $X = \mathbb{R}$, and the two infinite ray topologies on X, given by

$$\mathcal{T}_1 = \{(-\infty, a) \mid a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\} \quad \text{and} \quad \mathcal{T}_2 = \{(a, \infty) \mid a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}.$$

We proved that $\mathcal{T}_1$ is a topology in Worksheet 1, and one can give an analogous proof for $\mathcal{T}_2$. Then $U = (-\infty, 0) \in \mathcal{T}_1$ and $V = (0, \infty) \in \mathcal{T}_2$, but

$$U \cup V = \mathbb{R} \setminus \{0\} \notin \mathcal{T}_1 \cup \mathcal{T}_2,$$

since it is in neither $\mathcal{T}_1$ nor $\mathcal{T}_2$. Thus $\mathcal{T}_1 \cup \mathcal{T}_2$ is not a topology.

In fact, the intersection of U and V is also a problem: note that

$$U \cap V = \{0\} \notin \mathcal{T}_1 \cup \mathcal{T}_2.$$