Math 871 Topology I
Problem Set 2

Instructions: You will submit your work on gradescope as a pdf. If you choose to handwrite your work, please write neatly, and make sure everything is clearly legible in the scan. As a matter of good proof writing style, use complete sentences and correct grammar. You may use any result proven in class or in a homework problem, provided you reference it appropriately by either stating the result or its name. Please do not refer to theorems by their number in the course notes, as that can change.

You are strongly encouraged to work with your classmates and to discuss the problems with me. However, you should write your own final solutions. Never submit something for grading that you do not completely understand. See the course syllabus for the AI policy for this class.

1. Show that the topologies $\mathbb{R}_\ell$ and $\mathbb{R}_K$ on $\mathbb{R}$ are incomparable.

   Note: you can use without proof that these are topologies and that the bases for these topologies we gave in class are indeed bases.

2. Let $\mathscr{B}$ be a basis for a topology on X and let $\mathcal{T}$ be the topology generated by $\mathscr{B}$. Let $\{\mathcal{T}_\alpha\}_{\alpha \in I}$ be the collection of all topologies on X that contain $\mathscr{B}$. Show that

$$\mathcal{T} = \bigcap_{\alpha \in I} \mathcal{T}_\alpha.$$

3. Let $\mathscr{B}_1$ and $\mathscr{B}_2$ be bases for a topology on X.

   Note: we are *not* assuming $\mathscr{B}_1$ and $\mathscr{B}_2$ generate the same topology.

   (a) Show that the set

$$\mathscr{B} := \{B_1 \cap B_2 \mid B_1 \in \mathscr{B}_1 \text{ and } B_2 \in \mathscr{B}_2\}$$

   is also a basis for a topology on X.

   (b) Show that the topology $\mathcal{T}$ generated by $\mathscr{B}$ is the coarsest topology on X that contains both $\mathscr{B}_1$ and $\mathscr{B}_2$, meaning that all topologies $\mathcal{T}'$ with $\mathscr{B}_1 \subseteq \mathcal{T}'$ and $\mathscr{B}_2 \subseteq \mathcal{T}'$ satisfy $\mathcal{T} \subseteq \mathcal{T}'$.