Math 871 Topology I
Problem Set 1

1. Show that the topologies $\mathbb{R}_\ell$ and $\mathbb{R}_K$ on $\mathbb{R}$ are incomparable.

   Solution. Let $U = [-2,-1)$. Note that U is a basic open set in $\mathbb{R}_\ell$ with our usual choice of basis. However, we claim that U is not open in $\mathbb{R}_K$. Suppose, by contradiction, that U is an open set in $\mathbb{R}_K$. Then since $-2 \in U$, there exists a basic open set B in $\mathbb{R}_K$ such that $-2 \in B \subseteq U = [-2,-1)$. Then B must be of the form $B = (a,b)$ for some real numbers $a < b < -1$, and since $-2 \in B$, we must have $a < -2$. On the other hand, the fact that $a < -2$ contradicts the assumption that $(a,b) = B \subseteq [-2,-1)$. This shows that U is not open in $\mathbb{R}_K$.

   Now consider $V = (-1,1) \setminus K$, which is a basic open set in $\mathbb{R}_K$ under the usual choice of basis. We claim that V is not open in $\mathbb{R}_\ell$. Suppose, by contradiction, that V is in fact open in $\mathbb{R}_\ell$. Then there exists a basic open set $C = [a,b)$ in $\mathbb{R}_\ell$ such that $0 \in [a,b) \subseteq V$. In particular, note that we must have $b > 0$. On the other hand, there exists an integer $n \geqslant 1$ such that $0 < \frac{1}{b} < b$, and thus $[a,b) \cap K \neq \emptyset$. This contradicts the assumption that $[a,b) \subseteq V$, given that $V \cap K = \emptyset$, so we conclude that V is not open in $\mathbb{R}_K$.

2. Let $\mathscr{B}$ be a basis for a topology on X and let $\mathcal{T}$ be the topology generated by $\mathscr{B}$. Let $\{\mathcal{T}_\alpha\}_{\alpha \in I}$ be the collection of all topologies on X that contain $\mathscr{B}$. Show that

$$\mathcal{T} = \bigcap_{\alpha \in I} \mathcal{T}_\alpha.$$

   Solution. Since $\mathscr{B} \subseteq \mathcal{T}$ by definition of $\mathcal{T}$, we conclude that $\mathcal{T}$ is one of the topologies in $\{\mathcal{T}_\alpha\}_{\alpha \in I}$. Therefore,

$$\mathcal{T} \supseteq \bigcap_{\alpha} \mathcal{T}_\alpha.$$

   Let $U \in \mathcal{T}$. Then

$$U = \bigcup_{\beta \in J} B_\beta$$

   for some collection of $B_\beta \in \mathscr{B}$ for all $\beta \in J$. On the other hand,

$$B_\beta \in \mathscr{B} \subseteq \mathcal{T}_\alpha$$

   for all $\alpha \in I$ and $\beta \in J$. Fix α. Since $\mathcal{T}_\alpha$ is a topology and U is a union of open sets in T_α, we conclude that $U \in \mathcal{T}_\alpha$. But this holds for all α, so we conclude that every element in $\mathcal{T}$ is in $\mathcal{T}_\alpha$ for all α. Thus

$$\mathcal{T} \subseteq \bigcap_{\alpha} \mathcal{T}_\alpha.$$

   Together with the previous containment, this finishes our proof.

3. Let $\mathscr{B}_1$ and $\mathscr{B}_2$ each be bases for a topology on X.

   (a) Show that the set

$$\mathscr{B} := \{B_1 \cap B_2 \mid B_1 \in \mathscr{B}_1 \text{ and } B_2 \in \mathscr{B}_2\}$$

   is also a basis for a topology on X.

   Note: we are *not* assuming $\mathscr{B}_1$ and $\mathscr{B}_2$ generate the same topology.

   Solution. Let $x \in X$. Since $\mathscr{B}_1$ is a basis, there exists $B_1 \in \mathscr{B}_2$ such that $x \in B_1$. Since $\mathscr{B}_2$ is a basis, there exists $B_2 \in \mathscr{B}_2$ such that $x \in B_2$. Then $x \in B_1 \cap B_2 \in \mathscr{B}$.

   Let $U, V \in \mathscr{B}$. By definition, there exists $B_1, B_3 \in \mathscr{B}_1$ and $B_2, B_4 \in \mathscr{B}_2$ such that $U = B_1 \cap B_2$ and $V = B_3 \cap B_4$. Given

$$x \in U \cap V = B_1 \cap B_2 \cap B_3 \cap B_4,$$

   we have $x \in B_1$, $x \in B_2$, $x \in B_3$, and $x \in B_4$, so $x \in B_1 \cap B_3$ and $x \in B_2 \cap B_4$. Since $\mathscr{B}_1$ is a basis, there exists $C_1 \in \mathscr{B}_1$ such that $x \in C_1 \subseteq B_1 \cap B_3$. Since $\mathscr{B}_2$ is a basis, there exists $C_2 \in \mathscr{B}_2$ such that $x \in C_2 \subseteq B_2 \cap B_4$. Then

$$x \in C_1 \cap C_2 = (B_1 \cap B_3) \cap (B_2 \cap B_4) = (B_1 \cap B_2) \cap (B_3 \cap B_4) = U \cap V.$$

   Moreover, $C_1 \cap C_2 \in \mathscr{B}$. We conclude that $\mathscr{B}$ is a basis for a topology.

   (b) Show that the topology $\mathcal{T}$ generated by $\mathscr{B}$ is the coarsest topology on X that contains both $\mathscr{B}_1$ and $\mathscr{B}_2$, meaning that all topologies $\mathcal{T}'$ with $\mathscr{B}_1 \subseteq \mathcal{T}'$ and $\mathscr{B}_2 \subseteq \mathcal{T}'$ satisfy $\mathcal{T} \subseteq \mathcal{T}'$.

   Solution. First, we show that $\mathcal{T}$ is a topology containing both $\mathscr{B}_1$ and $\mathscr{B}_2$. Let $B \in \mathscr{B}_1$. Since $\mathscr{B}_2$ is a basis, for each $x \in B$ there exists $B_x \in \mathscr{B}_2$ such that $x \in B_x$, and thus $x \in B \cap B_x$. Note that

$$B = B \cap \left(\bigcup_{x \in B} \right) = \bigcup_{x \in B} (B \cap B_x).$$

   Since this is a union of open sets in $\mathcal{T}$, we conclude that $B \in \mathcal{T}$, and thus $\mathscr{B}_1 \subseteq \mathcal{T}$. A similar argument shows that $\mathscr{B}_2 \subseteq \mathcal{T}$.

   Let $\mathcal{T}'$ be a topology on X such that with $\mathscr{B}_1 \subseteq \mathcal{T}'$ and $\mathscr{B}_2 \subseteq \mathcal{T}'$. Let $B \in \mathscr{B}$. Then there exist $B_1 \in \mathscr{B}_1$ and $B_2 \in \mathscr{B}_2$ such that $B = B_1 \cap B_2$. By assumption, $\mathscr{B}_1 \subseteq \mathcal{T}'$ and $\mathscr{B}_2 \subseteq \mathcal{T}'$, and thus $B_1, B_2 \in \mathcal{T}'$. Since $\mathcal{T}'$ is a topology, we conclude that $B = B_1 \cap B_2 \in \mathcal{T}'$. Therefore, $\mathscr{B} \subseteq \mathcal{T}'$.

   By Exercise 2, $\mathcal{T}$ is contained in every topology that contains $\mathscr{B}$, and thus we must have $\mathcal{T} \subseteq \mathcal{T}'$.