

Math 871 Topology I

Problem Set 3

Instructions: You will submit your work on gradescope as a pdf. If you choose to handwrite your work, please write neatly, and make sure everything is clearly legible in the scan. As a matter of good proof writing style, use complete sentences and correct grammar. You may use any result proven in class or in a homework problem, provided you reference it appropriately by either stating the result or its name. Please do not refer to theorems by their number in the course notes, as that can change.

You are strongly encouraged to work with your classmates and to discuss the problems with me. However, you should write your own final solutions. Never submit something for grading that you do not completely understand. See the course syllabus for the AI policy for this class.

1. Consider the integers $\mathbb{Z}$ equipped with the finite complement topology. Compare the product topology on $\mathbb{Z} \times \mathbb{Z}$ with the finite complement topology on $\mathbb{Z} \times \mathbb{Z}$: determine if they are equal, and if not, if one is finer than the other, or if they are incomparable.
2. Consider the usual order on $\mathbb{N}$. Show that the dictionary topology on $\mathbb{N} \times \mathbb{N}$ is not the discrete topology.¹
3. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces.
 - (a) Show that if A is closed in X and B is closed in Y , then $A \times B$ is closed in $X \times Y$.
 - (b) Show that if A and B are nonempty, and $A \times B$ is closed in $X \times Y$, then A is closed in X .

Note: the analogous argument shows that B is closed in Y .

4. Let $X = \mathbb{R}_\ell \times \mathbb{R}_\ell$ equipped with the product topology, and let $Y = \{(x, -x) \mid x \in \mathbb{R}\}$. Determine the topology that Y inherits as a subspace of X .

Hint: this is a familiar topology.

¹In particular, this shows that the dictionary topology on $\mathbb{N} \times \mathbb{N}$ and the product topology on $\mathbb{N} \times \mathbb{N}$ are not the same!