Math 871 Topology I
Problem Set 3

1. Consider the integers $\mathbb{Z}$ equipped with the finite complement topology. Compare the product topology on $\mathbb{Z} \times \mathbb{Z}$ with the finite complement topology on $\mathbb{Z} \times \mathbb{Z}$: determine if they are equal, and if not, if one is finer than the other, or if they are incomparable.

   Solution. We claim that the product topology is strictly finer than the finite complement topology.

   To show that the product topology is finer than the finite complement topology, it suffices to show that every closed set in the finite complement topology is closed in the product topology. In the finite complement topology, the closed sets are $\mathbb{Z} \times \mathbb{Z}$ and all the finite subsets of $\mathbb{Z} \times \mathbb{Z}$. Since $\mathbb{Z} \times \mathbb{Z}$ and $\emptyset$ are closed in any topology, we only need to show that all nonempty finite subsets of $\mathbb{Z} \times \mathbb{Z}$ are closed in the product topology.

   First, consider a singleton $C = \{(a, b)\}$. Since $\{a\}$ and $\{b\}$ are closed in $\mathbb{Z}$ with the finite complement topology, $C = \{a\} \times \{b\}$ is also closed in $\mathbb{Z} \times \mathbb{Z}$ with the product topology, by problem 3. Since the finite union of closed sets is closed, we conclude that any nonempty finite subset of $\mathbb{Z} \times \mathbb{Z}$ is closed in the product topology. This completes the proof that the product topology is finer than the finite complement topology.

   To show it is strictly finer, consider $U = (\mathbb{Z} \setminus \{0\}) \times (\mathbb{Z} \setminus \{0\})$. While U is open for the product topology, it is a nonempty subset whose complement

   $$\mathbb{Z} \times \mathbb{Z} \setminus U = \mathbb{Z} \times \mathbb{Z} \setminus (\mathbb{Z} \setminus \{0\}) \times (\mathbb{Z} \setminus \{0\}) \supseteq \{(0, n) \mid n \in \mathbb{Z}, n \neq 0\}$$

   which is infinite. Therefore, $\mathbb{Z} \times \mathbb{Z} \setminus U$ is infinite, and thus U is not open in the finite complete topology.

2. Consider the usual order on $\mathbb{N}$. Show that the order topology on $\mathbb{N} \times \mathbb{N}$ is not the discrete topology.

   Solution. Let $x = (1, 0) \in \mathbb{N} \times \mathbb{N}$. We claim that the set $\{x\}$ is not open. To show that, consider the basis $B_<$ we gave in class, and consider any basic open set containing x. Since

   $$(0, 0) < (1, 0) < (1, 1),$$

   the point x is neither smallest nor largest in $\mathbb{N} \times \mathbb{N}$. Therefore, the only basic open sets that contain x are open intervals of the form $U = (a, b)$ for some $a < x < b$. Consider coordinates $a = (a_1, a_2)$ and $b = (b_1, b_2)$. Then

   $$a_1 \leqslant 1 \leqslant b_1.$$

   We will treat the case when $a_1 < b_1$ and when $a_1 = b_1$ separately.

   Suppose $a_1 = b_1$. Then we must have $a_1 = 1 = b_1$. Since $a < x < b$, this implies that

   $$a_2 < 0 < b_2.$$

However, $a_2 \in \mathbb{N}$, so this is impossible. We conclude that $a_1 < b_1$.

Now any element of the form

$$y = (y_1, y_2) \quad \text{with } y_1 = a_1 \text{ and } a_2 < y_2$$

satisfies

$$a_1 = y_1 \text{ and } a_2 < y_2 \implies a < y$$

and

$$y_1 = a_1 < b_1 \implies y < b,$$

and thus $y \in U$. There are infinitely many natural numbers $y_2 > a_2$, and thus infinitely many such y. We conclude that U is infinite, and in particular $\{x\}$ is not open.

3. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces.

   (a) Show that if A is closed in X and B is closed in Y, then $A \times B$ is closed in $X \times Y$.

   Solution. Let $U = X \times Y \setminus A \times B$ and $(x, y) \in U$. Then $x \notin A$ or $y \notin B$. Suppose $x \notin A$. Since A is closed in X, the complement $X \setminus A$ is open in X. Therefore, $V := (X \setminus A) \times Y$ is open in $X \times Y$. Moreover, note that $(x, y) \in (X \setminus A) \times Y$ and

   $$V \cap A \times B = (X \setminus A) \times Y \cap A \times B = \emptyset.$$

   Therefore, $V \subseteq U$. This proves that the open set V in $X \times Y$ satisfies

   $$(x, y) \subseteq V \subseteq U.$$

   Analogously, one can show that if $y \notin B$ then there exists an open set W such that

   $$(x, y) \subseteq W \subseteq U$$

   This shows that U is open in $X \times Y$, and hence that $A \times B$ is closed in $X \times Y$.

   Alternative Solution. Since A is closed in X, $X \setminus A$ is open in X. Since B is closed in Y, $Y \setminus B$ is open in Y. Then

   $$(X \setminus A) \times Y \quad \text{and} \quad X \times (Y \setminus B)$$

   are both open in $X \times B$, and thus its union is open. Moreover,

   $$\begin{aligned} ((X \setminus A) \times Y) \cup X\,(\times(Y \setminus B)) &= \{(x, y) \in X \times Y \mid x \notin A \text{ or } y \notin B\} \\ &= X \times Y \setminus \{(x, y) \in X \times Y \mid x \in A \text{ and } y \in B\} \\ &= X \times Y \setminus A \times B. \end{aligned}$$

   Therefore, $A \times B$ is closed.

(b) Show that if A and B are nonempty, and $A \times B$ is closed in $X \times Y$, then A is closed in X.

Note that analogously, we can conclude that B is closed in Y.

Solution. We will show that $X \setminus A$ is open. Let $X \setminus A$, meaning that $x \notin A$. Since B is nonempty, we can choose $y \in B$. Then $(x, y) \notin A \times B$. Recall that the set of all products of the form $U \times V$ with $U \in \mathcal{T}_X$ and $V \in \mathcal{T}_Y$ is a basis for the product topology. Since $X \times Y \setminus A \times B$ is open, we can find a basic open set $U \times V$ with $U \in \mathcal{T}_X$ and $V \in \mathcal{T}_Y$ such that

$$(x, y) \in U \times V \subseteq X \times Y \setminus A \times B.$$

Note that $U \times V \cap A \times B$.

Suppose that $U \cap A \neq \emptyset$, and let $a \in U \cap A$. Then $(a, y) \in U \times V \cap A \times B$, contradicting the fact that $U \times V \cap A \times B$. We conclude that $U \subseteq X \setminus A$. We have thus found an open set U such that

$$x \in U \subseteq X \setminus A.$$

Since this holds for all $x \in X \setminus A$, we conclude that $X \setminus A$ is open.

4. Let $X = \mathbb{R}_\ell \times \mathbb{R}_\ell$ equipped with the product topology, and let $Y = \{(x, -x) \mid x \in \mathbb{R}\}$. Determine the topology that Y inherits as a subspace of X.

Hint: this is a familiar topology.

Solution. We claim that the subspace topology $\mathcal{T}_Y$ induced by the topology on $\mathbb{R}_\ell \times \mathbb{R}_\ell$ is the discrete topology on Y. It suffices to show that single point sets are open in $\mathcal{T}_Y$, because if single point sets are open in Y, then so is any arbitrary union of single point sets, and every subset of Y can be expressed as the union of single point sets.

Let $y \in Y$, and note that y is represented by the ordered pair $(x, -x)$ for some $x \in \mathbb{R}$. Consider basic open sets in $\mathbb{R}_\ell$ given by $[x, x+1)$ and $[-x, -x+1)$. By the definition of the product topology, the set

$$U = [x, x+1) \times [-x, -x+1)$$

is open in $\mathbb{R}_\ell \times \mathbb{R}_\ell$. We claim that $\{y\} = U \cap Y$. Since

$$x \in [x, x+1) \quad \text{and} \quad -x \in [-x, -x+1),$$

then $y \in U \cap Y$. Given any $(a, b) \in U$ with $(a, b) \neq (x, -x)$, we must have $a > x$ or $a = x$. If $a > x$, then $-a < -x \leqslant b$, which means that $(a, b) \neq (a, -a)$ and thus $(a, b) \notin Y$. If $a = x$, then $b \neq -x$, so that $b > -x$ and $-b < x \leqslant a$, so $(-b, b) \neq (a, b)$ and thus $(a, b) \notin Y$. We conclude that $U \cap Y = \{y\}$, and by definition of the subspace topology $\mathcal{T}_Y$, the set $\{y\}$ is open in Y.

It follows by the above discussion that $\mathcal{T}_Y$ is the discrete topology on Y.