

Math 871 Topology I

Problem Set 6 solutions

1. Let X and Y be topological spaces and $f: X \rightarrow Y$ be an injective continuous function.

- (a) Prove that if Y is Hausdorff then so is X .

Solution. Consider any distinct points $a, b \in X$. Since f is injective, $f(a) \neq f(b)$. Since Y is Hausdorff, there are open neighborhoods U of $f(a)$ and V of $f(b)$ such that $U \cap V = \emptyset$. Since f is continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are open in X . Since $U \cap V = \emptyset$, we have $f^{-1}(U) \cap f^{-1}(V) = \emptyset$. Since $f(a) \in U$ and $f(b) \in V$, we have $a \in f^{-1}(U)$ and $b \in f^{-1}(V)$. This proves X is Hausdorff.

- (b) Does the converse of (a) hold?

Solution. No. For instance, take $Y = \{a, b\}$ with the indiscrete topology, and let $X = \{a\} \subseteq Y$. Note that X is Hausdorff under the subspace topology (which is also the unique topology on X : there are no two distinct points, so the Hausdorff condition holds trivially). Let $f: X \rightarrow Y$ be the inclusion map, with $f(a) = a$. Then f is injective. Moreover, we proved in class that the inclusion of a subspace is always continuous, so f is continuous. But Y is not Hausdorff, as the only open neighborhood of b is $Y \ni a$.

2. Let X be a Hausdorff space.

- (a) Show that if $f: X \rightarrow X$ is continuous, then

$$C := \{x \in X \mid f(x) = x\}$$

is closed.

Solution. Let $x \notin C$. By definition $f(x) \neq x$. Since X is Hausdorff, there exist open sets U and V such that $x \in U$, $f(x) \in V$, and $U \cap V = \emptyset$. Since f is continuous, $f^{-1}(V)$ is open. Therefore,

$$W := U \cap f^{-1}(V)$$

is open. Since $f(x) \in V$, then $x \in f^{-1}(V)$. Since $x \in U$, we conclude that $x \in W$.

We claim that $W \cap C = \emptyset$. Assume, by contradiction, that $z \in W \cap C$. Since $z \in C$, then $f(z) = z$. Since $z \in W \subseteq f^{-1}(V)$, $z = f(z) \in V$. But since $z \in W \subseteq U$, we would have $z \in U \cap V$, which contradicts the fact that $U \cap V = \emptyset$. We conclude that no such z exists and $W \cap C = \emptyset$.

In summary: for all $x \notin C$, we found an open set W with $x \in W$ and $W \cap C = \emptyset$, and thus C is closed.

- (b) Let $A \subseteq X$ have the subspace topology. Show that there exists a retraction $r: X \rightarrow A$, then A is a closed subset of X .

Solution. Let $f: X \rightarrow X$ be given by $f(x) = r(x)$. By construction, the image of f is A , and thus for all subsets $U \subseteq X$

$$f^{-1}(U) = r^{-1}(U \cap A).$$

If U is open in A , then $U \cap A$ is open in A by definition of the subspace topology, and thus $f^{-1}(U) = r^{-1}(U \cap A)$ is open in X since r is continuous. We conclude that f is continuous.

Now note that if $x \in A$, then $f(x) = r(x) = x$, and if $x \notin A$, then $f(x) \neq x$ since $f(x) \in A$ by construction. By (a),

$$\{x \in X \mid f(x) = x\} = \{x \in X \mid r(x) = x\} = A$$

is closed.

- (c) Does (b) hold if we do not assume that X is Hausdorff?

Solution. No. Let $X = \mathbb{Z}$ with the finite complement topology, and let $A = \mathbb{N}$. Note that A is not closed, as it is infinite and not the whole space. Moreover, we claim that the subspace topology on A is also the finite complement topology. Indeed, if $C \subsetneq A$ is closed in A , then $C = F \cap A$ for some finite set $F \subseteq \mathbb{Z}$, but then $C = F \cap \mathbb{N}$ is also finite. Conversely, all finite subsets of $\mathbb{N}$ are closed in $\mathbb{Z}$ and thus closed in $\mathbb{N}$. This shows that the nontrivial closed sets are the finite sets, and thus the subspace topology on A is the finite complement topology.

On the other hand, consider the function $v: \mathbb{Z} \rightarrow \mathbb{N}$ given by

$$v(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

We claim that v is continuous. Indeed, given any closed subset $C \subsetneq A$, C must be finite, say $C = \{a_1, \dots, a_n\}$, and

$$v^{-1}(C) = \{a_1, -a_1, \dots, a_n, -a_n\}$$

is also finite.¹ This shows that the preimage of a closed set is closed, and thus v is continuous.

Finally, note that $v(a) = a$ for all $a \in \mathbb{N}$, so v is indeed a retraction onto A .

Alternative solution. Consider $X = \{a, b\}$ with the trivial topology, and $A = \{a\}$. Note that X is not Hausdorff, as any point has X as its unique neighborhood. Moreover, A is not closed: its closure is X , since the only neighborhood of the unique point outside of A , b , is X , which intersects A nontrivially. Finally, the constant map $r: X \rightarrow A$ equal to A is continuous, since all constant maps are continuous, and since $r(a) = a$, this is a retraction of X onto A .

¹Silly note: if $a_i = 0$ for some i , the set above is still correct.