

Math 871 Topology I

Problem Set 7

1. The topological space given by $\mathbb{R}$ under the standard topology is a topological field: the following operations are all continuous when we view $\mathbb{R} \times \mathbb{R}$ with the product topology:

$$\begin{array}{llll} \mu: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} & s: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} & a: \mathbb{R} \rightarrow \mathbb{R} & i: \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R} \\ (x, y) \mapsto xy & (x, y) \mapsto x + y & x \mapsto -x & x \mapsto \frac{1}{x}. \end{array}$$

You may assume all four of these functions are continuous without proof.

- (a) Let X be any topological space and $f, g: X \rightarrow \mathbb{R}$ be continuous functions. Show that the following functions are all continuous:

$$\begin{array}{lll} f \cdot g: X \rightarrow \mathbb{R} & f + g: X \rightarrow \mathbb{R} & -f: X \rightarrow \mathbb{R} \\ x \mapsto f(x)g(x) & x \mapsto f(x) + g(x) & x \mapsto -f(x). \end{array}$$

Moreover, show that if $g(x) \neq 0$ for all $x \in X$, then the following is continuous:

$$\begin{array}{l} \frac{1}{g}: X \rightarrow \mathbb{R} \setminus \{0\} \\ x \mapsto \frac{1}{g(x)}. \end{array}$$

- (b) Show that the following is a closed subset of $\mathbb{R} \times \mathbb{R}$:

$$C = \left\{ \left(x, \frac{1}{x} \right) \mid x \neq 0 \right\}.$$

2. A function between topological spaces $f: X \rightarrow Y$ is

$$\begin{array}{ll} \text{an \textbf{open map} if} & U \text{ open in } X \implies f(U) \text{ open in } Y \\ \text{a \textbf{closed map} if} & C \text{ closed in } X \implies f(C) \text{ closed in } Y. \end{array}$$

- (a) Let $f: X \rightarrow Y$ be a function and $\mathcal{B}$ a basis for the topology on X . Show that if $f(U)$ is open for all $U \in \mathcal{B}$, then f is an open map.
- (b) Prove that if $\{X_\alpha\}_{\alpha \in I}$ is any collection of topological spaces, then under the product topology the projection maps

$$\pi_\beta: \prod_{\alpha \in I} X_\alpha \rightarrow X_\beta$$

are open maps for all $\beta \in I$.

- (c) Give an example showing that projection maps need not be closed.

Hint: You can find a counterexample using $\pi_1: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$.

3. Let X be any topological space and $f, g: X \rightarrow \mathbb{R}$ be continuous functions under the standard topology on $\mathbb{R}$.

- (a) Prove that $\{x \in X \mid f(x) \geq g(x)\}$ is a closed subset of X .
- (b) Prove that the function $h: X \rightarrow \mathbb{R}$ defined by

$$h(x) = \min\{f(x), g(x)\}$$

is continuous.

4. Let X and Y be topological spaces and let $f: X \rightarrow Y$ be any function.

- (a) Show that if f is continuous, then f sends convergent sequences to convergent sequences, meaning that for every sequence (x_n) in X converging to x , the sequence $(f(x_n))$ converges to $f(x)$.
- (b) We say X is **Fréchet-Urysohn** if for every subset $S \subseteq X$ and every $s \in \overline{S}$, there exists a sequence (s_n) of points in S converging to s . Show that if X is Fréchet-Urysohn and f sends convergent sequences to convergent sequences, then f is continuous.