

Math 871 Topology I

Problem Set 7 solutions

1. The topological space given by $\mathbb{R}$ with the standard topology is a topological field, meaning the following operations are continuous when $\mathbb{R} \times \mathbb{R}$ has the product topology:

$$\begin{array}{llll} \mu: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} & s: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} & a: \mathbb{R} \rightarrow \mathbb{R} & i: \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R} \\ (x, y) \mapsto xy & (x, y) \mapsto x + y & x \mapsto -x & x \mapsto \frac{1}{x}. \end{array}$$

You may assume all four of these functions are continuous without proof.

- (a) Let X be any topological space and $f, g: X \rightarrow \mathbb{R}$ be continuous functions. Show that the following functions are all continuous:

$$\begin{array}{lll} f \cdot g: X \rightarrow \mathbb{R} & f + g: X \rightarrow \mathbb{R} & -f: X \rightarrow \mathbb{R} \\ x \mapsto f(x)g(x) & x \mapsto f(x) + g(x) & x \mapsto -f(x). \end{array}$$

Moreover, show that if $g(x) \neq 0$ for all $x \in X$, then the following is continuous:

$$\begin{array}{l} \frac{1}{g}: X \rightarrow \mathbb{R} \setminus \{0\} \\ x \mapsto \frac{1}{g(x)}. \end{array}$$

Solution. By the universal property of products, the function $\langle f, g \rangle: X \rightarrow \mathbb{R} \times \mathbb{R}$ given by

$$\langle f, g \rangle(x) = (f(x), g(x))$$

is continuous, and it is the unique continuous function such that

$$\pi_1 \circ \langle f, g \rangle = f \text{ and } \pi_2 \circ \langle f, g \rangle = g.$$

If $g(x) \neq 0$ for all $x \in X$, then $g(X) \subseteq \mathbb{R} \setminus \{0\}$, and the function $\tilde{g}: X \rightarrow \mathbb{R} \setminus \{0\}$ induced by g by restricting the codomain is also continuous, by a theorem from class. Now note that

$$f \cdot g = \mu \circ \langle f, g \rangle \quad f + g = s \circ \langle f, g \rangle \quad -f = a \circ f \quad \frac{1}{g} = i \circ \tilde{g}$$

are all compositions of continuous functions, and thus continuous.

- (b) Show that the following is a closed subset of $\mathbb{R} \times \mathbb{R}$:

$$C = \left\{ \left(x, \frac{1}{x} \right) \mid x \in X \right\}.$$

Solution. Since $\mathbb{R}$ is Hausdorff, points are closed, and thus $\{1\} \subseteq \mathbb{R}$ is closed. Since μ is continuous, and

$$C = \mu^{-1}(\{1\})$$

is the preimage of a closed set by a continuous function, it is closed.

2. A function between topological spaces $f: X \rightarrow Y$ is

an **open map** if U open in $X \implies f(U)$ open in Y
 a **closed map** if C closed in $X \implies f(C)$ closed in Y .

(a) Let $f: X \rightarrow Y$ be a function and $\mathcal{B}$ a basis for the topology on X . Show that if $f(U)$ is open for all $U \in \mathcal{B}$, then f is an open map.

Solution. For any function f and any collection of sets $\{U_\alpha\}_{\alpha \in I}$, we have

$$f\left(\bigcup_{\alpha} U_{\alpha}\right) = \bigcup_{\alpha} f(U_{\alpha}).$$

Let U be any open set in X . We can write

$$U = \bigcup_{\alpha \in I} U_{\alpha}$$

for some collection of open sets $U_{\alpha} \in \mathcal{B}$. Thus

$$f\left(\bigcup_{\alpha} U_{\alpha}\right) = \bigcup_{\alpha} f(U_{\alpha})$$

is a union of open sets, and thus open.

(b) Prove that if $\{X_{\alpha}\}_{\alpha \in I}$ is any collection of topological spaces, then under the product topology the projection maps

$$\pi_{\beta}: \prod_{\alpha \in I} X_{\alpha} \rightarrow X_{\beta}$$

are open maps for all $\beta \in I$.

Solution. By a), we need only to consider a basis for the product topology, so it suffices to consider

$$U = \prod_{\alpha \in I} U_{\alpha}$$

where U_{α} is open in X_{α} for all α and for all but a finite number of α we have $U_{\alpha} = X_{\alpha}$. If $U = \emptyset$, then $\pi_{\alpha}(U) = \emptyset$ for all $\alpha \in I$. If $U \neq \emptyset$, then we claim that $\pi_{\alpha}(U) = U_{\alpha}$ for all $\alpha \in I$.

To check this, fix $\beta \in I$ and $u = (u_{\alpha})_{\alpha \in I} \in U$. For each $v \in U_{\beta}$, let

$$x = (x_{\alpha})_{\alpha \in I} \quad \text{where } x_{\alpha} = \begin{cases} v & \text{if } \alpha = \beta \\ u_{\alpha} & \text{if } \alpha \neq \beta. \end{cases}$$

Then $\pi_\beta(x) = v$, so $U_\beta \subseteq \pi_\beta(U)$. On the other hand, for any $s = (s_\alpha) \in U$ we have $\pi_\beta(s) = s_\beta \in U_\beta$, and thus $\pi_\beta(U) \subseteq U_\alpha$.

Note: If $U_\alpha \neq \emptyset$ for all $\alpha \in I$, then by the Axiom of Choice the product

$$U = \prod_{\alpha \in I} U_\alpha \neq \emptyset.$$

Equivalently, we can choose $u = (u_\alpha) \in U$. However, note that we did not need the Axiom of Choice to solve this problem.

- (c) Give an example showing that projection maps need not be closed.

Hint: You can find a counterexample using $\pi_1: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$.

Solution. Endow $\mathbb{R}$ with the standard topology on all copies, and consider $\pi_1: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$. By 1 b), the set

$$C = \left\{ \left(x, \frac{1}{x} \right) \mid x \neq 0 \right\} \subseteq \mathbb{R} \times \mathbb{R}$$

is closed. On the other hand,

$$\pi_1(C) = \{x \in \mathbb{R} \mid x \neq 0\}$$

is not closed, since its closure contains 0: for example, because nonempty open sets are infinite, and thus any open neighborhood of 0 contains points besides 0.

3. Let X be any topological space and $f, g: X \rightarrow \mathbb{R}$ be continuous functions under the standard topology on $\mathbb{R}$.

- (a) Prove that $\{x \mid f(x) \geq g(x)\}$ is a closed subset of X .

Solution. The subset $[0, \infty)$ of $\mathbb{R}$ is closed, as its complement $(-\infty, 0)$ is open. Moreover, since $f - g$ is continuous and

$$\{x \mid f(x) \geq g(x)\} = (f - g)^{-1}([0, \infty))$$

is the preimage of a closed set, this subset is closed.

- (b) Prove that the function $h: X \rightarrow \mathbb{R}$ defined by

$$h(x) = \min\{f(x), g(x)\}$$

is continuous.

Solution. Note that $X = A \cup B$, where

$$A = \{x \in X \mid f(x) \geq g(x)\} \quad \text{and} \quad B = \{x \in X \mid g(x) \geq f(x)\}.$$

By (a), both A and B are closed subsets of X . The restriction of a continuous function is continuous, so $g|_A$ is continuous on A and $f|_B$ is continuous on B . Moreover, these two functions agree on

$$A \cap B = \{x \mid f(x) = g(x)\}.$$

By the Pasting Lemma for closed sets, there is a unique continuous function $h: X \rightarrow \mathbb{R}$ such that $h|_A = g|_A$ and $h|_B = f|_B$. By the definition of A and B ,

$$h(x) = \min\{f(x), g(x)\} \text{ for all } x \in X.$$

4. Let X and Y be topological spaces and let $f: X \rightarrow Y$ be any function.

- (a) Show that if f is continuous, then f sends convergent sequences to convergent sequences, meaning that for every sequence (x_n) in X converging to x , the sequence $(f(x_n))$ converges to $f(x)$.

Solution. Let (x_n) be a sequence of points in X that converges to some point $x \in X$. Let U be any open neighborhood of $f(x)$ in Y . Since f is continuous and $f(x) \in U$, $f^{-1}(U)$ is an open neighborhood of x in X . Since (x_n) converges to x , there exists some N such that $x_n \in f^{-1}(U)$ for all $n \geq N$. Therefore, $f(x_n) \in U$, for all $n \geq N$. This proves that $(f(x_n))$ converges to $f(x)$.

- (b) We say X is **Fréchet-Urysohn** if for every subset $S \subseteq X$ and every $s \in \overline{S}$, there exists a sequence (s_n) of points in S converging to s . Show that if X is Fréchet-Urysohn and f sends convergent sequences to convergent sequences, then f is continuous.

Solution. Given any closed set C in Y , we will show its preimage $f^{-1}(C)$ is closed. If $f^{-1}(C) = \emptyset$, then it is closed. Suppose $f^{-1}(C) \neq \emptyset$, and let $x \in \overline{f^{-1}(C)}$. Since X is Fréchet-Urysohn, there exists a sequence of points $(x_n) \in f^{-1}(C)$ converging to x . Therefore, $(f(x_n))$ is a sequence in Y converging to $f(x)$. On the other hand, $(f(x_n))$ is a sequence of points in C . If $f(x) \notin C$, then we would be able to find an open neighborhood U of $f(x)$ such that $U \cap C = \emptyset$, and in particular U would contain none of the points in $(f(x_n))$. Therefore, $f(x) \in C$, so $x \in f^{-1}(C)$. We conclude that $f^{-1}(C)$ is closed and thus f is continuous.