

Math 871 Topology I

Problem Set 8 solutions

1. Consider the box topology on $\mathbb{R}^\omega$, and let A be the subset of $\mathbb{R}^\omega$ consisting of all the sequences whose terms are strictly positive:

$$A = \{x = (x_n)_{n \geq 1} \in \mathbb{R}^\omega \mid x_n > 0 \text{ for all } n \geq 1\}.$$

- (a) Let $z = (0)_{n \geq 1} \in \mathbb{R}^\omega$ be the constant sequence equal to zero. Show that $z \in \overline{A}$.

Solution. Suppose that

$$U = \prod_{n \geq 1} U_n$$

is a basic open neighborhood of z , with each $U_i = (a_i, b_i)$, so

$$U = (a_1, b_1) \times (a_2, b_2) \times \cdots.$$

Since $z \in U$, then $0 \in (a_i, b_i)$ for all i , so $b_i > 0$ and there exists $c_i \in (a_i, b_i)$ such that $0 < c_i < b_i$. Let

$$c = (c_1, c_2, \dots).$$

Then $c \in A \cap U$. We conclude that every open neighborhood of z intersects A nontrivially, and thus $z \in \overline{A}$.

- (b) Show that there is no sequence of elements of A converging to z .

Hint: given any sequence $(a_n)_{n \geq 1}$ of elements of A , let

$$a_n = (x_{n,1}, x_{n,2}, x_{n,3}, \dots) = (x_{n,i})_{i \geq 1}$$

and consider the open set

$$U = \prod_{n=1}^{\infty} (-x_{n,n}, x_{n,n}).$$

Solution. Note that for each i , we have $a_i \in A$, which means that its i th entry x_i^i is positive. Thus, the set U defined by

$$U = \prod_{n=1}^{\infty} (-x_{n,n}, x_{n,n})$$

is an open neighborhood of z in $\mathbb{R}^\omega$. Note that $x_{i,i} \notin (-x_{i,i}, x_{i,i})$, and so by construction, we have $a_i \notin U$. As this is true for all $i \geq 1$, it follows that the sequence $(a_n)_{n \geq 1}$ does not converge to z .

- (c) Show that $\mathbb{R}^\omega$ with the box topology is not metrizable.

Solution. We showed in class that if X is metrizable and $x \in \overline{A}$, then there exists a sequence of elements in A converging to x . By part (a) above, $z \in \overline{A}$, but by part (b), there does not exist a sequence of elements in A converging to z . We conclude that $\mathbb{R}^\omega$ is not metrizable.

2. Let $\bar{\rho}$ be the uniform metric on $\mathbb{R}^\omega$. Given $x = (x_n)_{n \geq 1} \in \mathbb{R}^\omega$ and $0 < \varepsilon \leq 1$, let

$$U(x, \varepsilon) = \prod_{n=1}^{\infty} (x_n - \varepsilon, x_n + \varepsilon).$$

Let $z = (0)_{n \geq 1} \in \mathbb{R}^\omega$ be the constant sequence equal to zero. Show that $U(z, 1)$ is not open in the uniform topology.

Solution. Consider the sequence given by

$$y = \left(0, \frac{1}{2}, \frac{2}{3}, \dots\right) = \left(\frac{n-1}{n}\right)_{n \geq 1}.$$

Note that $0 \leq y_n < 1$ for all n , so $y \in U(z, 1)$. We will show, however, that for all $\varepsilon > 0$, the uniform ball $B_{\bar{\rho}}(y, \varepsilon)$ is not contained in $U(z, 1)$. Fix $\varepsilon > 0$. Then there exists some integer N such that $N > \max\{\frac{1}{\varepsilon}, 1\}$. Define $x = (x_n)_{n \geq 1} \in \mathbb{R}^\omega$ by letting

$$x_n = \begin{cases} 1 & \text{if } n = N \\ y_n & \text{if } n \neq N. \end{cases}$$

Note that x and y differ only in their N th coordinates. Noting that

$$y_N - x_N = 1 - \frac{N-1}{N} = \frac{1}{N} \leq \varepsilon,$$

it follows that

$$\bar{\rho}(x, y) = |x_N - y_N| = \frac{1}{N} < \varepsilon,$$

and thus $x \in B_{\bar{\rho}}(y, \varepsilon)$. However, since $x_N = 1 \notin (-1, 1)$, we have $x \notin U(z, 1)$, and thus $B_{\bar{\rho}}(y, \varepsilon)$ is not a subset of $U(z, 1)$. It follows that $U(z, 1)$ is not open in the uniform topology.