Math 871 Topology I
Problem Set 9 solutions

1. Let $p\colon X \longrightarrow Y$ be a continuous map between topological spaces.

   (a) Show that if there is a continuous map $f\colon Y \to X$ such that $p \circ f = \mathrm{id}_Y$, then p is a quotient map.

   Solution. Let $y \in Y$. By assumption, $p(f(y)) = y$, and thus p is surjective. It remains to prove that for all subsets U of Y,

   $$U \text{ is open in } Y \iff p^{-1}(U) \text{ is open in } X.$$

   Since p is continuous, if U is open in Y, then $p^{-1}(U)$ is open in X. On the other hand, we always have

   $$U = (p \circ f)^{-1}(U) = f^{-1}(p^{-1}(U)).$$

   If $p^{-1}(U)$ is open in X, then $U = f^{-1}(p^{-1}(U))$ is open in Y since f is continuous. We have thus shown that p is a quotient map.

   (b) Prove that every retraction is a quotient map.

   Solution. Let $i\colon A \to X$ be the inclusion map, which we showed in class is continuous. Then $(r \circ i)(a) = r(a) = a$ by assumption, so that r and i satisfy the assumptions in part (a). We conclude that r is a quotient map.

2. Let $X = \mathbb{R}$ with the standard topology and define an equivalence relation on X by

   $$x \sim y \iff x - y \in \mathbb{Q}.$$

   Let $Y = X/\sim$ be the quotient of X by $\sim$. Show that Y is not Hausdorff.

   Hint: what is the topology on Y?

   Solution. We claim that Y has the trivial topology, in which the only open sets are Y and $\emptyset$. Once we show this, note that Y has at least two points, and therefore Y is not Hausdorff. Indeed, we have for example $[\pi] \neq [0]$, since $\pi - 0 = \pi \notin \mathbb{Q}$.

   Let $p\colon X \to Y$ the natural quotient map, and let us show that the topology in Y is trivial. Let U be any nonempty open set in Y, and let $V = p^{-1}(U)$. Since U is nonempty and p is surjective, V is also nonempty, so there exists some $x \in V$. Since V is open in $\mathbb{R}$ in the standard topology, there exists $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subseteq V$.

   We claim that $V = \mathbb{R}$. To show this, let $y \in \mathbb{R}$. By the density of the rationals in $\mathbb{R}$, since $\varepsilon > 0$ there exists a rational number c such that $x - y < c < x - y + \varepsilon$. It follows that $x < c + y < x + \varepsilon$, and thus $c + y \in V$. Since V is saturated with respect to p and $c + y \sim y$, we conclude that $y \in V$. This holds for all $y \in \mathbb{R}$, and thus $V = \mathbb{R}$. Since p is surjective,

   $$U = p(V) = p(\mathbb{R}) = Y.$$

   We have shown that Y is the unique nonempty open set in Y, and thus the topology on Y must be the trivial topology.

3. Consider the unit disk D^2, the subspace of $\mathbb{R}^2$ under the standard topology given by

$$D^2 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leqslant 1\}.$$

Define an equivalence relation $\sim$ on D^2 by

$$(x_1, y_1) \sim (x_2, y_2) \iff x_1 = x_2.$$

Show that $D^2/\sim$ is homeomorphic to the interval $[-1, 1]$ under the standard topology.

Solution. Define $f\colon D^2 \longrightarrow [-1, 1]$ by

$$f(x, y) = x.$$

We claim that f is continuous. To see that, first consider the map $s\colon D^2 \longrightarrow \mathbb{R}$ given by the composition of the inclusion of D^2 into $\mathbb{R}^2$ with the canonical projection $\mathbb{R}^2 \longrightarrow \mathbb{R}$ onto the first component, which are both continuous maps by a theorem from class. Since the composition of continuous maps is continuous, s is continuous. Viewing D^2 as a subset of $\mathbb{R}^2$, every $(x, y) \in D^2$ satisfies

$$x^2 + y^2 \leqslant 1 \implies -1 \leqslant x \leqslant 1 \implies \overline{f}(x, y) \in [-1, 1].$$

By a theorem from class, the restriction of the codomain of s to $[-1, 1]$ gives us a continuous map, and that map is precisely f. We conclude that f is continuous.

If $(x_1, y_1) \sim (x_2, y_2)$, then $x_1 = x_2$, and thus

$$f(x_1, y_1) = x_1 = x_2 = f(x_2, y_2).$$

By the universal property of the quotient, f induces a continuous map

$$\overline{f}\colon D^2/\sim \longrightarrow [-1, 1].$$

Consider the functions $g\colon [-1, 1] \longrightarrow D^2/\sim$ and $h\colon [-1, 1] \longrightarrow \mathbb{R}^2$ given by

$$g(x) = [(x, 0)] \quad \text{and} \quad h(x) = (x, 0).$$

We claim both g and h are continuous. We can justify that h is continuous via the universal property of the product: the inclusion i of $[-1, 1]$ into $\mathbb{R}$ is continuous and constant functions are continuous, and $h = (i, z)$ with z the constant equal to zero. On the other hand, the image of h is contained in D^2, so the restriction of h to $\widehat{h}\colon [-1, 1] \longrightarrow D^2$ is also continuous. Finally, g is the composition of $\widehat{h}$ with the canonical projection $p\colon D^2 \longrightarrow D^2/\sim$.

Now we claim that g is the inverse of $\overline{f}$. Indeed:

$$g \circ \overline{f}([x_1, x_2]) = g(x_1) = [(x_1, 0)] = [(x_1, x_2)]$$

and

$$\overline{f} \circ g(x) = \overline{f}([x, 0]) = x.$$