Math 871 Topology I
Worksheet 1 solutions

1. Let $(X, \mathcal{T})$ be a topological space, and let A be a subset of X. Suppose that for every $x \in A$, there exists an open set U such that $x \in U \subseteq A$. Show that A is an open set.

   Solution 1. By assumption, for each $x \in A$ there exists $U_x \in \mathcal{T}$ such that $x \in U_x \subseteq A$. In particular, note that
   $$A = \bigcup_{x \in A} \{x\} \subseteq \bigcup_{x \in A} U_x \subseteq A.$$
   Thus these containments must all be equalities, and A is a union of an arbitrary collection of open sets.

2. Prove that the finite complement topology on any set X is indeed a topology.

   Solution 2. By construction, $\emptyset$ is an open set. Moreover, X is open since its complement is empty (and thus finite).

   Let U and V be two open sets. If one of them is empty, then so is their intersection, which is thus open. Consider two nonempty open sets $U = X \setminus F$ and $V = X \setminus G$, so that F and G are finite. Then
   $$U \cap V = (X \setminus F) \cap (X \setminus G) = X \setminus (F \cup G).$$
   The union of finite sets is finite, and thus $F \cup G$ is finite. We conclude that $U \cap V$ is open.

   Finally, let $\{U_\alpha\}_{\alpha \in I}$ be a collection of open sets indexed by a set I. If one of them is empty, then it can be omitted from the collection without affecting the union, so we may assume that all the U_α are nonempty. Thus each U_α is of the form $U_\alpha = X \setminus F_\alpha$ for some finite F_α. Then
   $$U = \bigcup_{\alpha \in I} U_\alpha = \bigcup_{\alpha \in I} (X \setminus F_\alpha) = X \setminus \bigcap_{\alpha \in I} F_\alpha.$$
   The arbitrary intersection of finite sets is finite, and thus U is an open set.

3. Show that the **infinite ray topology** on $\mathbb{R}$, with open sets
   $$\mathcal{T} = \{(-\infty, a) \mid a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}.$$
   is indeed a topology.

   Hint: you may need to use the Completeness Axiom for $\mathbb{R}$, which says that

   Every nonempty, bounded above subset of $\mathbb{R}$ has a least upper bound.

Solution 3. First, note that X and $\emptyset$ are open sets by construction.

Moreover, given two open sets $(-\infty, a)$ and $(-\infty, b)$ for $a, b \in R$, note that

$$(-\infty, a) \cap (-\infty, b) = (-\infty, c) \qquad \text{where } c := \min\{a, b\}.$$

Therefore, the intersection of two open sets is open.

Consider a collection of open sets $\{U_\alpha\}_{\alpha \in I}$ indexed by a set I, and let

$$U = \bigcup_{\alpha \in I} U_\alpha.$$

If $U_\alpha = \emptyset$ for some α, then it can be omitted from the collection without affecting U. If $U_\alpha = \mathbb{R}$, then $U = \mathbb{R} \in \mathcal{T}$ is open. So assume $U_\alpha = (-\infty, r_\alpha)$ for all $\alpha \in I$. Now I could be the empty set, though in that case $U = \emptyset \in \mathcal{T}$ is open. Let us assume I is nonempty, and consider

$$S = \{r_\alpha \mid \alpha \in I\}$$

which is now necessarily nonempty since I is nonempty. We consider two cases: S is bounded above and S is not bounded above.

If S is not bounded above, then we claim that $U = \mathbb{R}$. On the one hand, $U \subseteq \mathbb{R}$ by construction. Let $x \in \mathbb{R}$. Since S is not bounded above, there is some α such that $x < r_\alpha$. But then $x \in U_\alpha \subseteq U$. Thus $U = \mathbb{R}$ is open.

Supposed S is bounded above. It is also nonempty, so the Completeness Axiom applies, and thus S has a least upper bound; let's call it ℓ. We claim that $U = (-\infty, \ell)$. Since $r_\alpha \leqslant \ell$, we have $U_\alpha \subseteq (-\infty, \ell)$ for all $\alpha \in I$, and hence $U \subseteq (-\infty, \ell)$. Let $x \in (-\infty, \ell)$. Then $x < \ell$ and hence x is not an upper bound of S. This means that there exists $\alpha \in I$ such that $x < r_\alpha$, and thus $x \in U_\alpha \subseteq U$. This proves that $U = (\infty, \ell)$ and hence $U \in \mathcal{T}$ is open.

4. Let X be a set and let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on X. We say that $\mathcal{T}$ is **finer** than $\mathcal{T}'$, or equivalently that $\mathcal{T}'$ is **coarser** than $\mathcal{T}$, if $\mathcal{T}' \subseteq \mathcal{T}$. Note that two topologies can also be **incomparable**. Compare the following topologies on $\mathbb{R}$:

   (a) $\mathcal{T}_1$ the standard topology.

   (b) $\mathcal{T}_2$ the finite complement topology.

   (c) $\mathcal{T}_3 = \{(-\infty, a) \mid a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$ the infinite ray topology.

   (d) $\mathcal{T}_4 = \{(a, \infty) \mid a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$.

   (e) $\mathcal{T}_5$ the discrete topology.

   (f) $\mathcal{T}_6$ the indiscrete topology.

Solution 4. Here is a hierarchy with the finest topology on top, where all lines indicate strict comparisons:

$$\begin{tikzcd}
& \mathcal{T}_5 \arrow[d, dash] & \\
& \mathcal{T}_1 \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\mathcal{T}_2 \arrow[dr, dash] & \mathcal{T}_3 \arrow[d, dash] & \mathcal{T}_4 \arrow[dl, dash] \\
& \mathcal{T}_6 &
\end{tikzcd}$$

Here are some notes on these comparisons:

- Note that $\mathcal{T}_1$ is always the finest topology on any set, and $\mathcal{T}_6$ is always the coarsest.
- The standard topology is strictly coarser than the indiscrete topology, since not every set is open in the standard topology. In fact, note that for example all open sets in the standard topology are infinite, so any nonempty finite set is open in $\mathcal{T}_5$ but not $\mathcal{T}_1$.
- The set $\{0\}$ is open in $\mathcal{T}_5$ but not in any of the other topologies.
- The set $(0, 1)$ is open in the standard topology but not in $\mathcal{T}_2$, $\mathcal{T}_3$, or $\mathcal{T}_4$.
- The set $\mathbb{R} \setminus \{0\} = (-\infty, 0) \cup (0, \infty)$ is open in $\mathcal{T}_2$ but not in $\mathcal{T}_3$ nor $\mathcal{T}_4$.
- The set $(-\infty, 0)$ is open in $\mathcal{T}_3$ but not in $\mathcal{T}_2$ nor $\mathcal{T}_4$.
- The set $(0, \infty)$ is open in $\mathcal{T}_4$ but not in $\mathcal{T}_2$ nor $\mathcal{T}_3$.
- If $U \in \mathcal{T}_2$ is nonempty, then $F = \mathbb{R} \setminus U$ is finite, say $F = \{a_1, \dots, a_n\}$. For every $x \notin F$, note that
$$\partial_x = \min_{1 \leqslant i \leqslant n} \{|x - a_i|\} > 0,$$
so we can choose a positive real number $\varepsilon_x < \partial_x$. Then $x \in (x - \varepsilon_x, x + \varepsilon_x) \subseteq U$, so U is an open set in the standard topology. This shows that $\mathcal{T}_2 \subseteq \mathcal{T}_1$.
- Let $U = (-\infty, a)$ be a nontrivial open set in $\mathcal{T}_3$. Note that for every $x \in U$, we have $x < a$, so we can choose a positive real number $\varepsilon < |a - x|$. Then $x \in (x - \varepsilon, x + \varepsilon) \subseteq U$, so U is open in the standard topology. This shows $\mathcal{T}_3 \subseteq \mathcal{T}_1$. Similarly, one can show that $\mathcal{T}_4 \subseteq \mathcal{T}_1$.

5. Show that the line with two origins is a topological space.

Solution 5. Note that the entire space X is open by definition. Let $\mathbf{0} \in \{0_1, 0_2\}$. I $U = \emptyset$, $U \setminus \{\mathbf{0}\} = U$ is open in $X \setminus \{\mathbf{0}\}$ (in any topology!).

Let U_1 and U_2 be two open sets. If one of them is empty, then $U_1 \cap U_2$ is empty and thus open. Assume that U_1 and U_2 are both nonempty. Then $U_1 \setminus \{\mathbf{0}\}$ and $U_2 \setminus \{\mathbf{0}\}$ are open in $X \setminus \{\mathbf{0}\}$, so

$$(U_1 \cap U_2) \setminus \{\mathbf{0}\} = (U_1 \setminus \{\mathbf{0}\}) \cap (U_2 \setminus \{\mathbf{0}\})$$

is open in $X \setminus \{\mathbf{0}\}$ since it is an intersection of two open sets.

Finally, consider a collection of open sets $\{U_\alpha\}_{\alpha \in I}$ indexed by a set I. If one of these is empty, then we can remove it from the collection without affecting the union, so let us assume that U_α is nonempty for all α. Then $U_\alpha \setminus \{\mathbf{0}\}$ is an open set in $X \setminus \{\mathbf{0}\}$ by definition, and thus

$$\left(\bigcup_{\alpha \in I} U_\alpha \right) \setminus \{\mathbf{0}\} = \bigcup_{\alpha \in I} (U_\alpha \setminus \setminus \{\mathbf{0}\})$$

is an arbitrary union of open sets of $X \setminus \{\mathbf{0}\}$, and thus open in $X \setminus \{\mathbf{0}\}$. We conclude that the union of these open sets is open in X.