Math 871 Topology I
Worksheet 2

1. Give an example of a topological space X and a nonempty subset $S \subsetneq X$ that is:

   (a) Open but not closed.

   (b) Closed but not open.

   (c) Not open and not closed.

   (d) **Clopen** = Open and closed.

2. Consider any totally ordered set X. Given any $a < b$ in X, show that $[a, b]$ is closed in the order topology.

3. Consider $X = \mathbb{R}$ as a topological space under the standard topology, and $Y = (0, 1)$ with the subspace topology, and $S = \left(0, \frac{1}{2}\right]$. Is S open in X? Is S closed in X? Is S open in Y? Is S closed in Y?

4. Consider $X = \mathbb{R}$ as a topological space under the finite complement topology. Consider $Y = [0, 2]$ as a subspace of X. For each of the following sets, answer the following questions: Is it open in X? Is it closed in X? Is it open in Y? Is it closed in Y?

   (a) $A = (1, 2]$.

   (b) $B = (0, 2]$.

   (c) $C = \{1\}$.

   (d) $D = [0, 2]$.

5. Consider the subspace $Y = [0, 1]$ of $\mathbb{R}$, and the usual order $<$ on $\mathbb{R}$. How does the order topology on Y compare to the subspace topology?

6. Consider $\mathbb{R}$ with the standard topology and the usual order. Let $Y = [0, 1) \cup \{2\} \subseteq \mathbb{R}$. Compare the subspace topology on Y with the order topology.

7. Let $\mathcal{B}_X$ be a basis for the topology on a topological space $(X, \mathcal{T}_X)$, and Y is any subset of X. Show that
$$\mathcal{B}_Y := \{U \cap Y \mid U \in \mathcal{B}_X\}$$
is a basis for the subspace topology on Y.

8. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and $A \subseteq X$ and $B \subseteq Y$ be subsets equipped with the corresponding subspace topology. Show that the product topology on $A \times B$ coincides with the subspace topology on $A \times B \subseteq X \times Y$ when $X \times Y$ is given the product topology.

9. Let X be a space with the finite complement topology, and let $Y \subseteq X$. Show that the subspace topology on X is the finite complement topology.

10. Let Y be a subspace of X and $A \subseteq Y$. Show that if A is closed in Y and Y is closed in X, then A is closed in X.

11. Show that if Y is a subspace of X, and A is a subset of Y, then the topology A in inherits as a subspace of Y is the same as the topology it inherits as a subspace of X.

12. Let X be a set a total order $<$, and $Y \subseteq X$. Consider X with the order topology.

    (a) Show that the subspace topology on Y is finer than the order topology.

    (b) Give an example showing that the subspace topology can be strictly finer than the order topology.

    (c) We say Y is **convex** if for all $a, b \in Y$, $a < b \implies (a, b)_X \subseteq Y$. Show that if Y is convex, then the order and subspace topologies on Y coincide.