Math 871 Topology I
Worksheet 2 solutions

1. Give an example of a topological space X and a nonempty subset $S \subsetneq X$ that is:

   (a) Open but not closed.

   Solution 1. In $\mathbb{R}$ with the standard topology, $(0,1)$ is open, but its complement is not open: any open neighborhood of 0 intersects $(0,1)$ nontrivially.

   (b) Closed but not open.

   Solution 2. In $\mathbb{R}$ with the standard topology, $\{0\}$ is closed, but not open: no open neighborhood of 0 is contained inside $\{0\}$.

   (c) Not open and not closed.

   Solution 3. In $\mathbb{R}$ with the standard topology, $(0,1]$ is neither closed nor open: any open neighborhood of 0 intersects $(0,1)$ nontrivially (so it is not closed) and no neighborhood of 1 is contained inside $(0,1]$ (so it is not open).

   (d) **Clopen** = Open and closed.

   Solution 4. In any set with two or more elements the discrete topology, we can take any nonempty proper subset. If this solution feels like cheating, here is another: let $X = (0,1) \cup (2,3)$ under the subspace topology induced by the standard topology on $\mathbb{R}$. Then $U = (0,1)$ is open in $\mathbb{R}$ and thus in X, and it is closed in X since
   $$X \setminus U = (2,3)$$
   is open in $\mathbb{R}$ and thus in X.

2. Consider any totally ordered set X. Given any $a < b$ in X, show that $[a,b]$ is closed in the order topology.

   Solution 5. Let
   $$(-\infty, a) := \{x \in X \mid x < a\}.$$
   If X has a smallest element $m \neq a$, then
   $$(-\infty, a) = [m, a)$$
   is open. If $m = a$, then $(-\infty, a) = \emptyset$ is still open. Otherwise,
   $$(-\infty, a) = \bigcup_{x<a} (x, a)$$
   is a union of basic open sets, and thus open. Similarly,
   $$(a, +\infty) := \{x \in X \mid a < x\}$$
   is open. Finally,
   $$X \setminus [a,b] = (-\infty, a) \cup (b, \infty)$$
   is open, and thus $[a,b]$ is closed.

3. Consider $X = \mathbb{R}$ as a topological space under the standard topology, and $Y = (0, 1)$ with the subspace topology, and $S = \left(0, \frac{1}{2}\right]$. Is S open in X? Is S closed in X? Is S open in Y? Is S closed in Y?

   Solution 6. It is closed in Y, as $S = \left[-1, \frac{1}{2}\right] \cap Y$ and $\left[-1, \frac{1}{2}\right]$ is closed in X. It is not closed in X: any open neighborhood of 0 contains $(-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$, and $(-\varepsilon, \varepsilon) \cap S = \emptyset$. It is not open in Y: any open neighborhood of $\frac{1}{2}$ in Y is of the form $U \cap Y$ for an open set U in X, and thus there exists $\varepsilon > 0$ such that

$$\left(\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon\right) \subseteq U.$$

   Let $\delta = \min\{\varepsilon, \frac{1}{2}\}$. Then

$$\left(\frac{1}{2} - \delta, \frac{1}{2} + \delta\right) \cap Y \subseteq U \cap Y.$$

   On the other hand, for all $\delta > 0$ we have

$$\left(\frac{1}{2} - \delta, \frac{1}{2} + \delta\right) \not\subseteq S.$$

   This shows that S is not open in Y. Since any subset of Y that is open in X is also open in Y, we conclude that S is also not open in X.

4. Consider $X = \mathbb{R}$ as a topological space under the finite complement topology. Consider $Y = [0, 2]$ as a subspace of X. For each of the following sets, answer the following questions: Is it open in X? Is it closed in X? Is it open in Y? Is it closed in Y?

   (a) $A = (1, 2]$.

   (b) $B = (0, 2]$.

   (c) $C = \{1\}$.

   (d) $D = [0, 2]$.

   Solution 7. As we will prove below, the subspace topology on Y is the finite complement topology. This allows us to answer quickly:

   Open in X: none of them.

   Open in Y: B, D.

   Closed in X: C.

   Closed in Y: C, D.

5. Consider the subspace $Y = [0,1]$ of $\mathbb{R}$, and the usual order $<$ on $\mathbb{R}$. How does the order topology on Y compare to the subspace topology?

Solution 8. The order topology has basis

$$\mathcal{B}_{\text{order}} = \{(a,b) \mid 0 \leqslant a < b \leqslant 1\} \cup \{[0,b) \mid 0 < b \leqslant 1\} \cup \{[a,1) \mid 0 \leqslant a < 1\}.$$

The order topology on $\mathbb{R}$ is the standard topology, with basis given by the open intervals. Thus the subspace topology on Y has basis

$$\begin{aligned}\mathcal{B}_{\text{sub}} &= \{(a,b) \cap [0,1] \mid a < b\} \\ &= \{(a,b) \mid 0 \leqslant a < b \leqslant 1\} \cup \{[0,b) \mid 0 < b \leqslant 1\} \cup \{[a,1) \mid 0 \leqslant a < 1\} \cup \{\emptyset\}.\end{aligned}$$

The only difference between the two bases is the empty set, which is an open set in any topology. We conclude that the two basis are the same, and thus the two topologies agree.

6. Consider $\mathbb{R}$ with the standard topology and the usual order. Let $Y = [0,1) \cup \{2\} \subseteq \mathbb{R}$. Compare the subspace topology on Y with the order topology.

Solution 9. As we will see below, the subspace topology is always finer than the order topology. On the other hand, we claim that is this case the subspace topology is strictly finer. Indeed,

$$\{2\} = \left(\frac{3}{2}, 3\right) \cap Y$$

is open in the subspace topology, but on the other hand 2 is the largest value in Y, so any basic open set for the order topology in Y containing 2 has to be of the form $(a,2]$ for some $a < 2$ in Y, but that implies that $a < 1$, and for any such a we have

$$(a,2]_Y = (a,1) \cup \{2\} \neq \{2\}.$$

7. Let $\mathcal{B}_X$ be a basis for the topology on a topological space $(X, \mathcal{T}_X)$, and Y is any subset of X. Show that

$$\mathcal{B}_Y := \{U \cap Y \mid U \in \mathcal{B}_X\}$$

is a basis for the subspace topology on Y.

Solution 10. See the notes.

8. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and $A \subseteq X$ and $B \subseteq Y$ be subsets equipped with the corresponding subspace topology. Show that the product topology on $A \times B$ coincides with the subspace topology on $A \times B \subseteq X \times Y$ when $X \times Y$ is given the product topology.

Solution 11. See the notes.

9. Let X be a space with the finite complement topology, and let $Y \subseteq X$. Show that the subspace topology on X is the finite complement topology.

Solution 12. The closed sets in X are X, $\emptyset$, and all the finite subsets of X. If a nonempty set $C \subsetneq Y$ is closed in Y, then $C = D \cap Y$ for some closed set D in X.

If $C \subseteq Y$ is closed in Y, then $C = D \cap Y$ for some closed set D in X. If moreover $C \neq Y$ is nonempty, then D must be nonempty and $D \neq X$. Then D has to be finite, and thus $C = D \cap Y$ has to be finite.

Conversely, if $C \subsetneq Y$ is finite, then C is closed in X, and thus $C = C \cap Y$ is closed in Y. We conclude that the closed sets in Y are Y, $\emptyset$, and the finite subsets of Y. Therefore, the subspace topology on Y is the finite complement topology.

10. Let Y be a subspace of X and $A \subseteq Y$. Show that if A is closed in Y and Y is closed in X, then A is closed in X.

Solution 13. Suppose that A is closed in Y and Y is closed in X. Then there exists a set B, closed in X, such that $A = B \cap Y$. Since the intersection of closed sets is again closed, and B and Y are closed in X, it follows that A is closed in X as well.

11. Show that if Y is a subspace of X, and A is a subset of Y, then the topology A in inherits as a subspace of Y is the same as the topology it inherits as a subspace of X.

Solution 14. Let $\mathcal{T}$ be the subspace topology on A induced by the topology on Y and $\mathcal{T}'$ is the subspace topology on A induced by the topology on X. Let $\mathcal{T}_X$ be the topology of the space X, and let $\mathcal{T}_Y$ be the subspace topology on Y (as a subspace of X). By definition of subspace topology,

$$\mathcal{T}_Y := \{U \cap Y \mid U \in \mathcal{T}_X\}.$$

and

$$\mathcal{T} := \{A \cap U \mid U \in \mathcal{T}_Y\} \qquad \mathcal{T}' := \{A \cap V \mid V \in \mathcal{T}_X\}.$$

We want to show that $\mathcal{T} = \mathcal{T}'$. Let $W \in \mathcal{T}$. Then $W = A \cap U$ for some $U \in \mathcal{T}_Y$. Moreover, $U = Y \cap V$ for some $V \in \mathcal{T}_X$. Combining these gives us

$$W = A \cap (Y \cap V) = (A \cap Y) \cap V.$$

Since A is a subset of Y, then $A \cap Y = A$, and so $W = A \cap V$ with $V \in \mathcal{T}_X$. Therefore, $W \in \mathcal{T}'$, which proves that $\mathcal{T} \subseteq \mathcal{T}'$.

Let $W' \in \mathcal{T}'$. By definition, $W' = A \cap V$ for some $V \in \mathcal{T}_X$. Again combining this with the fact that $A = A \cap Y$, we get

$$W' = (A \cap Y) \cap V = A \cap (Y \cap V),$$

and

$$Y \cap V \in \mathcal{T}_Y.$$

Therefore, $W' \in \mathcal{T}$, which proves that $\mathcal{T}' \subseteq \mathcal{T}$.

12. Let X be a set a total order $<$, and $Y \subseteq X$. Consider X with the order topology.

(a) Show that the subspace topology on Y is finer than the order topology.

Solution 15. It suffices to check that all the elements in a basis for the order topology are open in the subspace topology. Given $a < b \in Y$,

$$(a,b)_Y = \{x \in Y \mid a < x < b\} = \{x \in X \mid a < x < b\} \cap Y = (a,b)_X \cap Y.$$

Thus $(a,b)_Y$ is open in the subspace topology.
Suppose Y has a smallest element, meaning there exists $m \in Y$ such that $m \leqslant y$ for all $y \in Y$. If m is also the smallest element in X, then for each $b \in Y$ such that $m < b$ we have

$$[m,b)_Y = \{x \in Y \mid m \leqslant x < b\} = \{x \in X \mid m \leqslant x < b\} \cap Y = [m,b)_X \cap Y.$$

If m is not the smallest element in X, then there exists $z \in X$ such that $z < m$, and for each $b \in Y$ with $m < b$ we have

$$[m,b)_Y = (z,b)_X \cap Y.$$

In both cases, we showed that $[m,b)_Y$ is open in the subspace topology. Similarly, we can show that if Y has a largest element M, then $(a,M]_Y$ is open in the subspace topology for all $a \in Y$ with $a < M$.

(b) Give an example showing that the subspace topology can be strictly finer than the order topology.

Solution 16. We already gave an example in this worksheet: $X = \mathbb{R}$ with the usual order and $Y = [0,1) \cup \{2\}$.

(c) We say Y is **convex** if for all $a, b \in Y$, $a < b \implies (a,b)_X \subseteq Y$. Show that if Y is convex, then the order and subspace topologies on Y coincide.

Solution 17. Again, it suffices to check that all the elements in a basis for the subspace topology are open in the order topology. So we need to show that given any $a < b$ in X, $(a,b) \cap Y$ is open in Y in the order topology.
If $(a,b) \cap Y = \emptyset$, then it is open in Y (in any topology). Suppose that $(a,b) \cap Y \neq \emptyset$. Then there exists $c \in Y$ with $a < c < b$.
If there exists $z \in Y$ with $z < a$, then by convexity since $z < a < c$ and $a, c \in Y$, we conclude that $a \in (z,c)_X \subseteq Y$. Thus if $a \notin Y$, then $a < y$ for all $y \in Y$. Similarly, if $b \notin Y$, the convexity gives us that $y < b$ for all $y \in Y$.

- Case 1: $a, b \in Y$. Then by convexity, $(a,b) \subseteq Y$, and thus $(a,b) = (a,b) \cap Y$ is open in the subspace topology.
- Case 2: $a, b \notin Y$. Then $a < y < b$ for all $y \in Y$, so

$$(a,b) \cap Y = Y$$

is open in Y (in any topology!).

- Case 3: $a \in Y$, $b \notin Y$. If Y has a largest element M, then note that $M \neq a$, since there exists $c \in Y$ with $a < c < b$. By convexity $(a, M) \subseteq Y$, and

$$(a, b) \cap Y = (a, M] \cap Y = (a, M]_Y$$

is open in the order topology. If Y has no largest element, then

$$(a, b) \cap Y = \bigcup_{\substack{x \in Y \\ a < x}} (a, x).$$

In this case, convexity gives us that for every $x \in Y$ with $a < x$, $(a, x) \subseteq Y$, and thus $(a, b) \cap Y$ is a union of open sets in the order topology of Y.

- Case 4: $a \notin Y$, $b \in Y$. Analogous to case 3.